Oliver M. O'Reilly

Engineering Dynamics

A Primer

2nd Edition

Springer

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Second edition

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To the memory of my father,
John “Jackie” O’Reilly (1935–2009)

Preface

Preface to Second Edition

The possibility of publishing an affordable electronic version of this book was the primary reason for writing the second edition. I have also taken the opportunity to reformat, rearrange, and update the original edition. New examples have been added and for most of the examples, such as the rolling disk and the particle on a cone, more detailed analyses have been presented. Several explanations in the text have, I hope, been improved upon. Where appropriate, numerical results for the motion of individual systems have been included.

It is a pleasure to take this opportunity to once again thank Achi Dosanjh at Springer-Verlag in New York for her editorial support. Charles Taylor's daughter, Mary Anne Taylor Urry, kindly gave me permission to use the photo shown in Figure 10.1. I am also grateful to Joanna Chang, Nur Adila Faruk Senan, and Daniel Kawano for their help with this edition and to many others for their comments on the first edition. Although I hope that there are no typographical errors in this edition, in the event that my hopes are dashed, I would welcome hearing about them by email at oreilly@berkeley.edu. An updated errata will be posted on my University of California at Berkeley homepage.

All of the author royalties from the sales of this book will be donated to the United Nations Children's Fund (UNICEF).

Berkeley, December 2009

Oliver M. O'Reilly
www.me.berkeley.edu/faculty/oreilly/

Preface to First Edition

This primer is intended to provide the theoretical background for the standard undergraduate course in dynamics. This course is usually based on one of the following texts: Bedford and Fowler [6], Beer and Johnston [7], Hibbeler [36], Meriam and Kraige [48], Riley and Sturges [63], and Shames [69], among others. Although most teachers will have certain reservations about these texts, there appears to be a general consensus that the selection of problems each of them presents is an invaluable and essential aid for studying and understanding dynamics.

I use Meriam and Kraige [48] when teaching such a course, which is referred to as ME104 at the University of California at Berkeley (UCB). However, I have found that the gap between the theory presented in the aforementioned texts and the problems I wished my students to solve was too large. As a result, I prepared my own set of notes on the relevant theory, and I used Meriam and Kraige [48] as a problem and homework resource. This primer grew out of these notes. Its content was also heavily influenced by three other courses that I teach: one on rigid body dynamics, one on Lagrangian mechanics, and another on Hamiltonian mechanics.[1] Because I use the primer as a supplement, I have only included a set of brief exercises at the end of each chapter. Furthermore, dimensions of physical quantities and numerical calculations are not emphasized in the primer because I have found that most students do not have serious problems with these matters.

This primer is intended for three audiences: students taking an undergraduate engineering dynamics course, graduate students needing a refresher in such a course, and teachers of such a course. For the students, I hope that this primer succeeds in providing them with a succinct account of the theory needed for the course, an exploration of the limitations of such a course, and a message that the subject at hand can be mastered with understanding and not rote memorization of formulae. For all of these audiences, an appendix provides the notational and presentational correspondences between the chapters in this primer and the aforementioned texts. In addition, each chapter is accompanied by a summary section.

I have noticed an increased emphasis on "practical" problems in engineering dynamics texts. Although such an emphasis has its merits, I think that the most valuable part of an education is the evolution and maturation of the student's thinking abilities and thought processes. With this in mind, I consider the development of the student's analytical skills to be paramount. This primer reflects my philosophy in this respect.

The material in this primer is not new. I have merely reorganized some classical thoughts and theories on the subject in a manner that suits an undergraduate engineering dynamics course. My sources are contained in the references section at the end of this primer. Apart from the engineering texts listed above, the works of Beatty [5], Casey [14, 16], and Synge and Griffith [78] had a significant influence on my exposition.

I have also included some historical references and comments in this primer in the hopes that some students may be interested in reading the original work. Most of the historical information in the primer was obtained from Scribner's *Dictionary of Scientific Biography*. I heartily recommend reading the biographies of Euler, Kepler, Leibniz, and others contained in this wonderful resource.

Finally, for two reasons I have tried wherever possible to outline the limitations of what is expected from a student. First, some students will decide to extend their

[1] These courses are referred to as ME170, ME175, and ME275 in the UCB course catalog.

knowledge beyond these limitations, and, second, it gives a motivation to the types of questions asked of the student.

My perspective on dynamics has been heavily influenced by both the continuum mechanics and dynamics communities. I mention in particular the writings and viewpoints of Jim Casey, Jim Flavin, Phil Holmes, Paul M. Naghdi, Ronald Rivlin, and Clifford Truesdell. I owe a large debt of gratitude to Jim Casey both for showing me the intimate relationship between continuum mechanics and dynamics, and for supporting my teaching here at Berkeley.

The typing of this primer using LaTeXwould not have been possible without the assistance of Bonnie Korpi, Laura Cantú, and Linda Witnov. Laura helped with the typing of Chapters 7, 8, 9, and 10. Linda did the vast majority of the work on the remaining chapters. Her patience and cheerful nature in dealing with the numerous revisions and reorganizations was a blessing for me. David Kramer was a copy-reader for the primer, and he provided valuable corrections to the final version of the primer. The publication of this primer was made possible by the support of Achi Dosanjh at Springer-Verlag. Achi also organized two sets of helpful reviews. Several constructive criticisms made by the anonymous reviewers have been incorporated, and I would like to take this opportunity to thank them.

Many of my former students have contributed directly and indirectly to this primer. In particular, Tony Urry read through an earlier draft and gave numerous insightful comments on the presentation. I have also benefited from numerous conversations with my former graduate students Tom Nordenholz, Jeffrey Turcotte, and Peter Varadi. As I mentioned earlier, this primer arose from my lecture notes for ME104. My interactions with the former students in this course have left an indelible impression on this primer.

Finally, I would like to thank my wife, Lisa, my parents, Anne and Jackie, and my siblings, Séamus and Sibéal, for their support and encouragement.

Berkeley, December 2000 *Oliver M. O'Reilly*

Contents

Part I
Dynamics of a Single Particle

Chapter 1
Elementary Particle Dynamics

TOPICS

Here, we cover the basics on kinematics and kinetics of particles and discuss three ubiquitous examples. We conclude with a discussion of Euler's first law (which is also known as Newton's second law or the balance of linear momentum). Our treatment of dynamics makes extensive use of vector calculus. For the interested student, a summary of the needed results from vector calculus is presented in Appendix A.

1.1 An Example

Consider the example of a particle that is launched into the air from a point with an initial velocity. During the subsequent motion of the particle, it is subject to a gravitational force and a drag force. The gravitational force is constant whereas the magnitude of the drag force is proportional to the cube of the speed of the particle

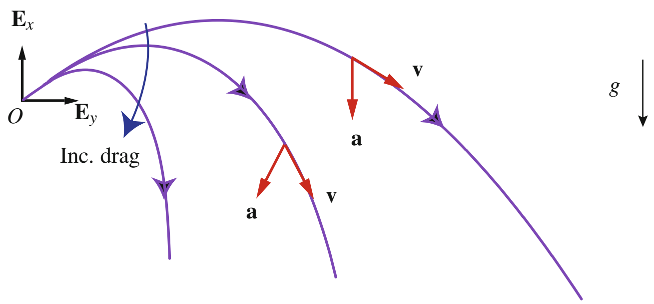

Fig. 1.1 *Representative paths of a particle falling under gravity and subject to a drag force of* $-mk\|\mathbf{v}\|^2\mathbf{v}$. *Some examples of the velocity,* $\mathbf{v}$, *and acceleration,* $\mathbf{a}$, *vectors of the particle are also shown.*

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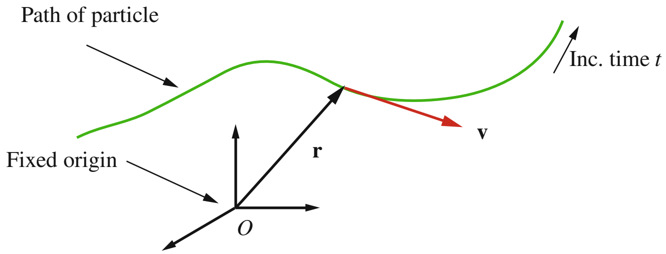

Fig. 1.2 *Some kinematical quantities pertaining to a particle and its motion.*

and opposes the motion of the particle. For a given set of initial conditions, sample trajectories for the particle can be seen in Figure 1.1. After reading this chapter, you should be able to show that a representation for the drag force is $-mC_d \|\mathbf{v}\|^2 \mathbf{v}$ where the coefficient of drag C_d is a constant, to know how to formulate the differential equations governing the motion of the particle, and to understand the analytical solution to the resulting equations when the drag force is absent.

1.2 Kinematics of a Particle

Consider a particle moving in a three-dimensional space $\mathbb{E}^3$. The position vector $\mathbf{r}$ of the particle relative to a fixed origin O as a function of time is denoted by the function $\mathbf{r}(t)$. That is, given a time t, the location of the particle is determined by the value $\mathbf{r} = \mathbf{r}(t)$ (see Figure 1.2). Varying t, $\mathbf{r}(t)$ defines the motion and the path $\mathscr{C}$ of the particle. This path in many cases coincides with a specific curve, for example, a particle moving on a circular ring or a particle in motion on a circular helix. Otherwise, the particle is either free or in motion on a surface.

The (absolute) velocity vector $\mathbf{v}$ of the particle can be determined by differentiating $\mathbf{r}(t)$ with respect to time t:

$$\mathbf{v} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \lim_{\triangle t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

The speed v of the particle is given by the magnitude of the velocity vector: $v = \|\mathbf{v}\|$. We often denote the time derivative of a function by a superposed dot, for example, $\mathbf{v} = \dot{\mathbf{r}}$. The (absolute) acceleration vector $\mathbf{a}$ of the particle is determined by differentiating the (absolute) velocity vector with respect to time:

$$\mathbf{a} = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \lim_{\triangle t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}.$$

To calculate the distance traveled by a particle along its path, it is convenient to introduce the arc-length parameter s. This parameter is defined by

$$\frac{ds}{dt} = \|\mathbf{v}\|.$$

Clearly, $ds/dt \geq 0$ is the speed of the particle. Integrating this relationship one finds that

$$s(t) - s_0 = \int_{t_0}^{t} \frac{ds}{dt}(\tau)d\tau = \int_{t_0}^{t} \sqrt{\mathbf{v}(\tau)\cdot\mathbf{v}(\tau)}\, d\tau.$$

It should be noted that $s(t) - s_0$ is the distance traveled by the particle along its path $\mathscr{C}$ during the time interval $t - t_0$. Also, $s(t_0) = s_0$, where t_0 and s_0 are initial conditions. You should also notice that we use a dummy variable τ when performing the integration.[1]

Often, instead of using t to parametrize the motion of the particle, one uses the arc-length parameter s. We show several examples of this parametrization in Chapter 3. The motion will, in general, be a different function of s than it is of t. To distinguish these functions, we denote the motion as a function of t by $\mathbf{r}(t)$ and the motion as a function of s is denoted by $\hat{\mathbf{r}}(s)$. Provided $\dot{s}$ is never zero, these functions provide the same value of $\mathbf{r}$: $\mathbf{r}(t) = \hat{\mathbf{r}}(s(t))$.[2]

With the above proviso in mind, one has the relations

$$\begin{aligned}
s &= s(t) = s_0 + \int_{t_0}^{t} \sqrt{\mathbf{v}(\tau)\cdot\mathbf{v}(\tau)}\, d\tau, \\
\mathbf{r} &= \mathbf{r}(t) = \hat{\mathbf{r}}(s(t)), \\
\mathbf{v} &= \frac{d\mathbf{r}(t)}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt}, \\
\mathbf{a} &= \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{r}}{ds}\frac{d^2s}{dt^2}.
\end{aligned}$$

At this stage, we have not used a particular coordinate system, so all of the previous results are valid for any coordinate system. For most of the remainder of this primer we use three different sets of orthonormal bases: Cartesian $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$, cylindrical polar $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$, and the Serret-Frenet triad $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$. Which set one uses depends on the problem of interest. Knowing which one to select is an art, and to acquire such experience is very important.[3]

---

[1] It is necessary to use a dummy variable τ as opposed to the variable t when evaluating this integral because we are integrating the magnitude of the velocity as τ varies between t_0 and t. If we take the derivative with respect to t of the integral, then, using the fundamental theorem of calculus, we would find, as expected, that $\dot{s}(t) = \|\mathbf{v}\|$. Had we not used the dummy variable τ but rather t to perform the integration, then the derivative of the resulting integral with respect to t would not yield $\dot{s}(t) = \|\mathbf{v}(t)\|$.

[2] Here, we are invoking the inverse function theorem of calculus. If $\dot{s}$ were zero, then the particle would be stationary (i.e., s would be constant), but time would continue increasing, so there would not be a one-to-one correspondence between s and t.

[3] In other words, the more problems one examines, the better.

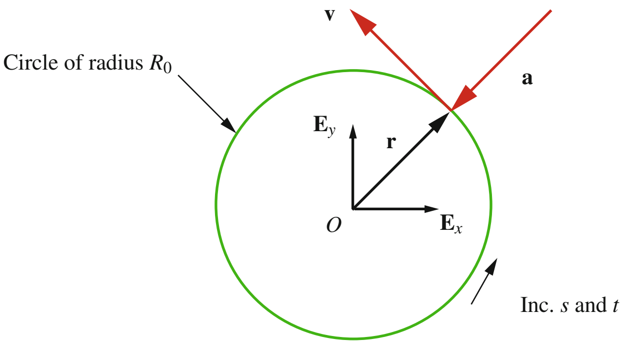

Fig. 1.3 *Path of a particle performing a circular motion.*

1.3 A Circular Motion

We now elucidate the preceding developments with a simple example. Suppose the position vector of a particle has the representation

$$\mathbf{r} = \mathbf{r}(t) = R_0\left(\cos(\omega t)\mathbf{E}_x + \sin(\omega t)\mathbf{E}_y\right),$$

where ω is a positive constant and R_0 is greater than zero. In Figure 1.3, the path of the particle is shown. You should notice that the path of the particle is a circle of radius R_0 that is traversed in a counterclockwise direction.[4] The question we seek to answer here is what are $\mathbf{v}(t)$, $\mathbf{a}(t)$, $s(t)$, $t(s)$, $\hat{\mathbf{v}}(s)$, and $\hat{\mathbf{a}}(s)$?

First, let's calculate $\mathbf{v}(t)$ and $\mathbf{a}(t)$:

$$\begin{aligned}\mathbf{v} &= \dot{\mathbf{r}} = R_0\omega(-\sin(\omega t)\mathbf{E}_x + \cos(\omega t)\mathbf{E}_y),\\ \mathbf{a} &= \ddot{\mathbf{r}} = \dot{\mathbf{v}} = -R_0\omega^2(\cos(\omega t)\mathbf{E}_x + \sin(\omega t)\mathbf{E}_y) = -\omega^2\mathbf{r}.\end{aligned}$$

Next,

$$s(t) - s_0 = \int_{t_0}^{t} \sqrt{\mathbf{v}(\tau)\cdot\mathbf{v}(\tau)}\, d\tau = \int_{t_0}^{t} \sqrt{\omega^2 R_0^2} d\tau = R_0\omega\,(t - t_0)\,.$$

Hence,

$$t(s) = t_0 + \frac{1}{R_0\omega}\,(s - s_0)\,.$$

---

[4] Later on, we hope sooner rather than later, you should revisit this problem using the cylindrical polar coordinates, $r = R_0$ and $\theta = \omega t$, and establish the forthcoming results using cylindrical polar basis vectors. This coordinate system is discussed in Chapter 2.

The last formula allows us to write down the following results with a minimum of effort using $\mathbf{v}(t)$ and $\mathbf{a}(t)$:

$$
\begin{aligned}
\mathbf{v} &= \hat{\mathbf{v}}(s) = R_0\omega\left(-\sin\left(\frac{s-s_0}{R_0}+\omega t_0\right)\mathbf{E}_x + \cos\left(\frac{s-s_0}{R_0}+\omega t_0\right)\mathbf{E}_y\right),\\
\mathbf{a} &= \hat{\mathbf{a}}(s) = -R_0\omega^2\left(\cos\left(\frac{s-s_0}{R_0}+\omega t_0\right)\mathbf{E}_x + \sin\left(\frac{s-s_0}{R_0}+\omega t_0\right)\mathbf{E}_y\right)\\
&= -\omega^2\hat{\mathbf{r}}(s).
\end{aligned}
$$

Alternatively, one could use the expression for $t(s)$ to determine $\mathbf{r}$ as a function of s and then differentiate with respect to t to obtain the desired functions.

1.4 Rectilinear Motions

In this section, we consider the motion of a particle along a straight line. We take $\mathbf{E}_x$ to be parallel to this line and the vector $\mathbf{c}$ to be a constant. Then,

$$
\begin{aligned}
\mathbf{r} &= \mathbf{r}(t) = x(t)\mathbf{E}_x + \mathbf{c},\\
\mathbf{v} &= \mathbf{v}(t) = \frac{dx}{dt}\mathbf{E}_x = v(t)\mathbf{E}_x,\\
\mathbf{a} &= \mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{E}_x = a(t)\mathbf{E}_x.
\end{aligned}
$$

It is important to note that

$$
\frac{ds}{dt} = \left|\frac{dx}{dt}\right|,
$$

so unless $\dot{x} > 0$ or $\dot{x} < 0$, x and s cannot be easily interchanged.

The material that follows should be familiar to you from other courses. Furthermore, graphical interpretations of the forthcoming results are readily available. Essentially, they involve relating t, x, v, and a. There are three cases to consider.

1.4.1 Given Acceleration as a Function of Time

Suppose one knows $a(t)$; then $v(t)$ and $x(t)$ can be determined by integrating $a(t)$:

$$
\begin{aligned}
v(t) &= v(t_0) + \int_{t_0}^{t} a(u)du,\\
x(t) &= x(t_0) + \int_{t_0}^{t} v(\tau)d\tau = x(t_0) + \int_{t_0}^{t}\left(v(t_0) + \int_{t_0}^{\tau} a(u)du\right)d\tau.
\end{aligned}
$$

You should notice that u and τ in these expressions are dummy variables.

1.4.2 Given Acceleration as a Function of Speed

The next case to consider is when $a = \bar{a}(v)$ and one is asked to calculate $\bar{x}(v)$ and $\bar{t}(v)$. The former task is achieved by noting that $dt = dv/a$, whereas the latter task is achieved by using the identities

$$a = \frac{dv}{dt} = \left(\frac{dv}{dx}\right)\left(\frac{dx}{dt}\right) = v\frac{dv}{dx}.$$

In summary,

$$\bar{t}(v) = \bar{t}(v_0) + \int_{v_0}^{v} \frac{1}{\bar{a}(u)} du, \qquad \bar{x}(v) = \bar{x}(v_0) + \int_{v_0}^{v} \frac{u}{\bar{a}(u)} du.$$

We remark that the identity $a = vdv/dx$ is very useful and is featured in Sections 3.5.4 and 5.3.

1.4.3 Given Acceleration as a Function of Placement

The last case to consider is when $a = \hat{a}(x)$ is known and one seeks $\hat{v}(x)$ and $\hat{t}(x)$. Again, the former result is calculated using the identity $a = vdv/dx$, and the latter is calculated using the identity $dt = dx/v$:

$$\hat{v}^2(x) = \hat{v}^2(x_0) + 2\int_{x_0}^{x} \hat{a}(u)du, \qquad \hat{t}(x) = \hat{t}(x_0) + \int_{x_0}^{x} \frac{du}{\hat{v}(u)}.$$

1.5 Kinetics of a Particle

Consider a particle of constant mass m. Let $\mathbf{F}$ denote the resultant external force acting on the particle, and let $\mathbf{G} = m\mathbf{v}$ be the linear momentum of the particle. Euler's first law[5] (also known as Newton's second law[6] or the balance of linear momentum) postulates that

$$\mathbf{F} = \frac{d\mathbf{G}}{dt} = m\mathbf{a}.$$

[5] Leonhard Euler (1707–1783) made enormous contributions to mechanics and mathematics. We follow C. Truesdell (see Essays II and V in [79]) in crediting $\mathbf{F} = m\mathbf{a}$ to Euler. As noted by Truesdell, these differential equations can be seen on pages 101–105 of a 1749 paper by Euler [24]. Truesdell's essays also contain copies of certain parts of a related seminal paper [25] by Euler that was published in 1752.

[6] Isaac Newton (1642–1727) wrote his second law in Volume 1 of his famous *Principia* in 1687 as follows: *The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.* (Cf. page 13 of [52].)

It is crucial to note that **a** is the absolute acceleration vector of the particle. In the following we write this equation with respect to several sets of basis vectors. For instance, with respect to a (right-handed) Cartesian basis, the vector equation $\mathbf{F} = m\mathbf{a}$ is equivalent to three scalar equations:

$$F_x = ma_x = m\frac{d^2x}{dt^2},$$
$$F_y = ma_y = m\frac{d^2y}{dt^2},$$
$$F_z = ma_z = m\frac{d^2z}{dt^2},$$

where

$$\mathbf{F} = F_x\mathbf{E}_x + F_y\mathbf{E}_y + F_z\mathbf{E}_z, \qquad \mathbf{a} = a_x\mathbf{E}_x + a_y\mathbf{E}_y + a_z\mathbf{E}_z.$$

1.5.1 Action and Reaction

When dealing with the forces of interaction between particles, a particle and a rigid body, and rigid bodies, we also invoke Newton's third law: "For every action there is an equal and opposite reaction." For example, consider a particle moving on a surface. From this law, the force exerted by the surface on the particle is equal in magnitude and opposite in direction to the force exerted by the particle on the surface.

1.5.2 The Four Steps

There are four steps to solving problems using $\mathbf{F} = m\mathbf{a}$:

1. Pick an origin and a coordinate system, and then establish expressions for $\mathbf{r}, \mathbf{v}$, and **a**.
2. Draw a free-body diagram.
3. Write out $\mathbf{F} = m\mathbf{a}$.
4. Perform the analysis.

These four steps will guide you through most problems. We amend them later on, in an obvious way, when dealing with rigid bodies. If you follow them, they will help you with homework and exams.

One important point concerns the free-body diagram. This is a graphical summary of the external forces acting on the particle. It does not include any accelerations. Here, in contrast to some other treatments, it is used only as an easy visual check on one's work.

1.6 A Particle Under the Influence of Gravity

Consider a particle of mass m that is launched with an initial velocity $\mathbf{v}_0$ at $t = 0$. At this instant, $\mathbf{r} = \mathbf{r}_0$. During the subsequent motion of the particle it is under the influence of a vertical gravitational force $-mg\mathbf{E}_y$. In SI units, g is approximately 9.81 meters per second per second (m s^{-2}). One is asked to determine the path $\mathbf{r}(t)$ of the particle.

The example of interest is a standard projectile problem. It also provides a model for the motion of the center of mass of many falling bodies where the influence of drag forces is ignored. For example, it is a model for a vehicle falling through the air. To determine the motion of the particle predicted by this model, we follow the four aforementioned steps. After completing the analysis, we show how a drag force can be included.

1.6.1 Kinematics

For this problem it is convenient to use a Cartesian coordinate system. One then has the representations

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z, \qquad \mathbf{a} = \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y + \ddot{z}\mathbf{E}_z.$$

1.6.2 Forces

The sole force acting on the particle is gravity, so $\mathbf{F} = -mg\mathbf{E}_y$ and the free-body diagram is trivial. It is shown in Figure 1.4.

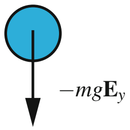

Fig. 1.4 *Free-body diagram of a particle in a gravitational field.*

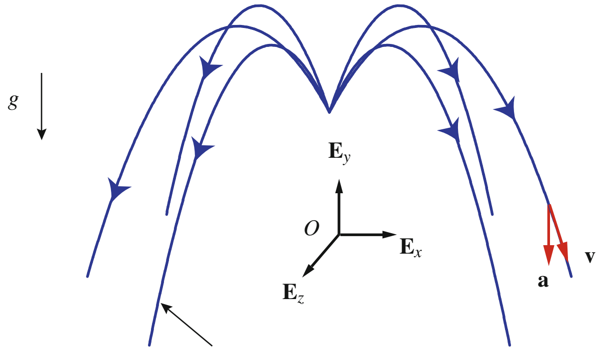

Fig. 1.5 *Representative paths of a particle falling under the influence of gravity.*

1.6.3 Balance Law

From $\mathbf{F} = m\mathbf{a}$, we obtain three second-order ordinary differential equations:

$$
\begin{aligned}
m\ddot{x} &= 0, \\
m\ddot{y} &= -mg, \\
m\ddot{z} &= 0.
\end{aligned}
$$

1.6.4 Analysis

The final step in solving the problem involves finding the solution to the previous differential equations that satisfies the given initial conditions:

$$
\begin{aligned}
\mathbf{r}_0 &= \mathbf{r}(t=0) = x_0\mathbf{E}_x + y_0\mathbf{E}_y + z_0\mathbf{E}_z, \\
\mathbf{v}_0 &= \mathbf{v}(t=0) = \dot{x}_0\mathbf{E}_x + \dot{y}_0\mathbf{E}_y + \dot{z}_0\mathbf{E}_z.
\end{aligned}
$$

The differential equations in question have simple solutions:

$$
\begin{aligned}
x(t) &= \dot{x}_0 t + x_0, \\
y(t) &= -\frac{1}{2}gt^2 + \dot{y}_0 t + y_0, \\
z(t) &= \dot{z}_0 t + z_0.
\end{aligned}
$$

Hence, the motion of the particle can be written in a compact form:

$$
\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{1}{2}gt^2\mathbf{E}_y.
$$

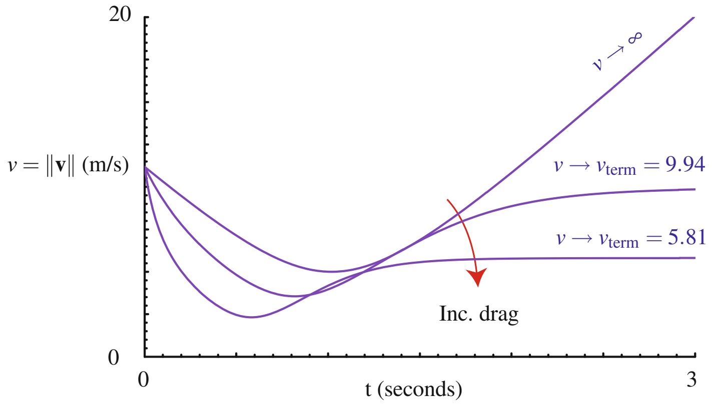

Fig. 1.6 *The speed $v = \|\mathbf{v}\|$ of the particle as a function of time for the three particle paths shown in Figure 1.1.*

By specifying particular sets of initial conditions, the results for the motion of the particle apply to numerous special cases. Several representative examples of these paths where $\mathbf{r}_0$ is fixed and $\mathbf{v}_0$ is varied are shown in Figure 1.5.

We now consider the modifications needed to the analysis if a drag force $\mathbf{F}_D$ were included. You may recall that this example was discussed in Section 1.1. We suppose that the drag force is proportional to v^3 and opposes the motion of the particle. The latter assumption implies that $\mathbf{F}_D$ is parallel to $\mathbf{v}/\|\mathbf{v}\|$. As $v = \|\mathbf{v}\|$, we infer that $\mathbf{F}_D$ has the representation

$$\mathbf{F}_D = -mC_d v^2 \mathbf{v},$$

where C_d is a nonnegative constant. With the drag force introduced, $\mathbf{F} = -mg\mathbf{E}_y - mC_d v^2 \mathbf{v}$. Using a balance of linear momentum, we arrive at the equations of motion:

$$m\ddot{x} = -mC_d v^2 \dot{x}, \qquad m\ddot{y} = -mg - mC_d v^2 \dot{y}, \qquad m\ddot{z} = -mC_d v^2 \dot{z},$$

where

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$

Solutions to the equations of motion for the values $C_d = 0, 0.01, 0.05$ and the initial conditions $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = 5\mathbf{E}_x + 10\mathbf{E}_y$ are shown in Figure 1.1. The speed of the particle as a function of time for these three cases is shown in Figure 1.6.

You may have noticed from Figure 1.1 that as the drag coefficient C_d increases the trajectory deviates more and more from the parabolic form of the drag-free case. In the presence of drag, the trajectory asymptotes to a rectilinear vertical motion and the speed of the particle asymptotes to the terminal velocity v_{term}. To compute the terminal velocity, we set the accelerations to zero in the equations of motion and solve for the resulting velocity vector:

$$\mathbf{v}_{\text{term}} = -v_{\text{term}}\mathbf{E}_y = -\left(\frac{g}{C_d}\right)^{1/3}\mathbf{E}_y.$$

Clearly, the terminal speed v_{term} is independent of the initial velocity vector of the particle and tends to ∞ as $C_d \to 0$ (see Figure 1.6).

1.7 Summary

In this chapter, several definitions of kinematical quantities pertaining to a single particle were presented. In particular, the position vector $\mathbf{r}$ relative to a fixed origin was defined. This vector defines the path of the particle. Furthermore, the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors were defined:

$$\mathbf{v} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}, \qquad \mathbf{a} = \mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2}.$$

These vectors can also be defined as functions of the arc-length parameter s: $\mathbf{v} = \hat{\mathbf{v}}(s)$ and $\mathbf{a} = \hat{\mathbf{a}}(s)$. Here, s is defined by integrating the differential equation $\dot{s} = \|\mathbf{v}\|$. The parameter s can be used to determine the distance traveled by the particle along its path. Using the chain rule, it was shown that

$$\mathbf{v} = \dot{s}\frac{d\mathbf{r}}{ds}, \qquad \mathbf{a} = \dot{s}^2\frac{d^2\mathbf{r}}{ds^2} + \ddot{s}\frac{d\mathbf{r}}{ds}.$$

Two special cases of the aforementioned results were discussed in Sections 1.3 and 1.4. First, in Section 1.3, the kinematics of a particle moving in a circular path was discussed. Then, in Section 1.4, the corresponding quantities pertaining to rectilinear motion were presented.

The balance of linear momentum $\mathbf{F} = m\mathbf{a}$ was then introduced. This law relates the motion of the particle to the resultant force $\mathbf{F}$ acting on the particle. In Cartesian coordinates, it can be written as three scalar equations:

$$\begin{aligned} F_x &= ma_x = m\ddot{x}, \\ F_y &= ma_y = m\ddot{y}, \\ F_z &= ma_z = m\ddot{z}, \end{aligned}$$

where

$$\mathbf{F} = F_x\mathbf{E}_x + F_y\mathbf{E}_y + F_z\mathbf{E}_z, \qquad \mathbf{a} = a_x\mathbf{E}_x + a_y\mathbf{E}_y + a_z\mathbf{E}_z.$$

In order to develop a helpful problem-solving methodology, a series of four steps was introduced. These steps are designed to provide a systematic framework to help guide you through problems. To illustrate the steps, a well-known projectile problem was discussed in Section 1.6.

1.8 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

1.1. Why are the time derivatives of $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ all equal to zero?

1.2. Suppose that you are given a vector as a function of s: $\mathbf{f}(s)$. Why do you need to know how s depends on time t in order to determine the derivative of $\mathbf{f}$ as a function of time?

1.3. Consider a particle of mass m that lies at rest on a horizontal surface. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the particle. Draw a free-body diagram of the particle. If one follows the four steps, then why is it a mistake to write $mg\mathbf{E}_y$ for the normal force instead of $N\mathbf{E}_y$, where N is unknown, in the free-body diagram?

1.4. The motion of a particle is such that its position vector $\mathbf{r}(t) = 10\mathbf{E}_x + 10t\mathbf{E}_y + 5t\mathbf{E}_z$ (meters). Show that the path of the particle is a straight line and that the particle moves along this line at a constant speed of $\sqrt{125}$ meters per second. Furthermore, show that the force $\mathbf{F}$ needed to sustain this motion is $\mathbf{0}$.

1.5. The motion of a particle is such that its position vector $\mathbf{r}(t) = 3t\mathbf{E}_x + 4t\mathbf{E}_y + 10\mathbf{E}_z$ (meters). Show that the path of the particle is a straight line and that the particle moves along this line at a constant speed of 5 meters per second. Using this information, show that the arc-length parameter s is given by $s(t) = 5(t - t_0) + s_0$. Finally, show that the particle moves 50 meters along its path every 10 seconds.

1.6. The motion of a particle is such that its position vector $\mathbf{r}(t) = 10\cos(n\pi t)\mathbf{E}_x + 10\sin(n\pi t)\mathbf{E}_y$ (meters). Show that the particle is moving on a circle of radius 10 meters and describes a complete circle every $2/n$ seconds. If the particle has a mass of 2 kilograms, then what force $\mathbf{F}$ is needed to sustain this motion?

1.7. To model the free-fall of a ball of mass m, the ball is modeled as a particle of the same mass. Suppose the particle is dropped from the top of a 100 meter high building. Following the steps discussed in Section 1.6, show that it takes $\sqrt{200/9.81}$ seconds for the ball to reach the ground. Furthermore, show that it will hit the ground at a speed of $\sqrt{1962}$ meters per second.[7]

1.8. A projectile is launched at time $t_0 = 0$ seconds from a location $\mathbf{r}(t_0) = \mathbf{0}$. The initial velocity of the projectile is $\mathbf{v}(t_0) = v_0\cos(\alpha)\mathbf{E}_x + v_0\sin(\alpha)\mathbf{E}_y$. Here, v_0 and α are constants. During its flight, a vertical gravitational force $-mg\mathbf{E}_y$ acts on the projectile. Modeling the projectile as a particle of mass m, show that its path is a parabola:

$$y(x) = -\left(\frac{g}{2v_0^2\cos^2(\alpha)}\right)x^2 + \tan(\alpha)x.$$

[7] This speed is equal to 99.09 miles per hour.

Why is this result not valid when $\alpha = \pm\pi/2$?

1.9. Consider the ball discussed in Exercise 1.7, and now suppose that the particle is subject to a drag force whose magnitude is proportional to the speed of the particle. Argue that the drag force has the representation $-mk\mathbf{v}$ in this case, and show that the terminal speed of the ball is $v_{\text{term}} = g/k$.

Chapter 2
Particles and Cylindrical Polar Coordinates

TOPICS

Here, we discuss the cylindrical polar coordinate system and how it is used in particle mechanics. This coordinate system and its associated basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$ are vital to understand and practice.

It is a mistake to waste your time memorizing formulae here. Instead, focus on understanding the material. You will repeat it countless times which will naturally develop the ability to derive the results from scratch.

2.1 The Cylindrical Polar Coordinate System

Consider the pendulum system shown in Figure 2.1. Here, a particle of mass m is attached by an inextensible string of length L to a fixed point O. Assuming that the string remains taut, then the distance from O to the particle remains constant: $\|\mathbf{r}\| =$

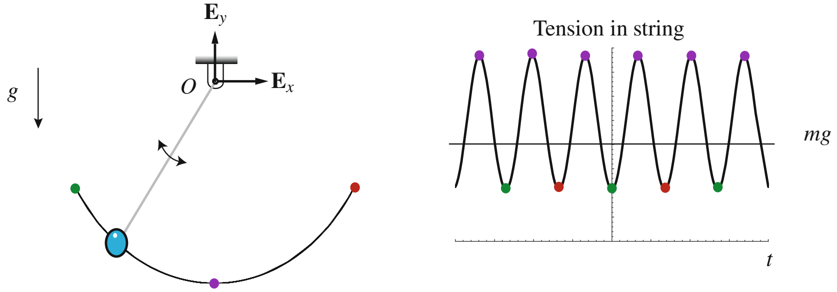

Fig. 2.1 *An example of the motion of a pendulum. The behavior of the tension T in the string during this motion is also shown.*

O.M. O'Reilly, *Engineering Dynamics: A Primer*, DOI 10.1007/978-1-4419-6360-4_2,

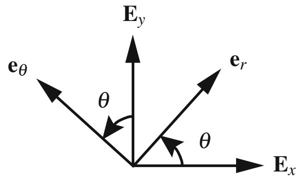

Fig. 2.2 *The unit vectors* $\mathbf{e}_r$ *and* $\mathbf{e}_\theta$.

L. This system is a prototypical example of a situation where a polar coordinate system can be effectively used.

To define a cylindrical polar coordinate system $\{r, \theta, z\}$, we start with a Cartesian coordinate system $\{x, y, z\}$ for the three-dimensional space $\mathbb{E}^3$. Using these coordinates, we define r, θ, and z as

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad z = z.$$

The coordinate $r \geq 0$. Apart from the points $\{x, y, z\} = \{0, 0, z\}$, given r, θ, and z, we can uniquely determine x, y, and z:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z.$$

Here, θ is taken to be positive in the counterclockwise direction.

If we now consider the position vector $\mathbf{r}$ of a point in this space, we have, as always,

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z.$$

We can write this position vector using cylindrical polar coordinates by substituting for x and y in terms of r and θ:

$$\mathbf{r} = r\cos(\theta)\mathbf{E}_x + r\sin(\theta)\mathbf{E}_y + z\mathbf{E}_z.$$

Before we use this representation to establish expressions for the velocity and acceleration vectors, it is convenient to introduce the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_x \\ \mathbf{E}_y \\ \mathbf{E}_z \end{bmatrix}.$$

Two of these vectors are shown in Figure 2.2.

Note that $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$ are orthonormal and form a right-handed basis[1] for $\mathbb{E}^3$. You should also be able to see that

$$\begin{bmatrix} \mathbf{E}_x \\ \mathbf{E}_y \\ \mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{E}_z \end{bmatrix}.$$

Because $\mathbf{e}_r = \mathbf{e}_r(\theta)$ and $\mathbf{e}_\theta = \mathbf{e}_\theta(\theta)$, these vectors change as θ changes:

$$\frac{d\mathbf{e}_r}{d\theta} = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y = \mathbf{e}_\theta,$$
$$\frac{d\mathbf{e}_\theta}{d\theta} = -\cos(\theta)\mathbf{E}_x - \sin(\theta)\mathbf{E}_y = -\mathbf{e}_r.$$

It is crucial to note that θ is measured positive in the counterclockwise direction.

Returning to the position vector $\mathbf{r}$, it follows that

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z = \underbrace{r\cos(\theta)\mathbf{E}_x + r\sin(\theta)\mathbf{E}_y}_{=r\mathbf{e}_r} + z\mathbf{E}_z = r\mathbf{e}_r + z\mathbf{E}_z.$$

Furthermore, because $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$ is a basis, we then have, for any vector $\mathbf{b}$, that

$$\mathbf{b} = b_r\mathbf{e}_r + b_\theta\mathbf{e}_\theta + b_z\mathbf{E}_z = b_x\mathbf{E}_x + b_y\mathbf{E}_y + b_z\mathbf{E}_z.$$

It should be clear that $b_r = \mathbf{b}\cdot\mathbf{e}_r$, $b_\theta = \mathbf{b}\cdot\mathbf{e}_\theta$, and $b_z = \mathbf{b}\cdot\mathbf{E}_z$.

2.2 Velocity and Acceleration Vectors

Consider a particle moving in space: $\mathbf{r} = \mathbf{r}(t)$. We recall that

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{E}_z = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z.$$

As the particle is in motion, its coordinates are functions of time: $x = x(t)$, $y = y(t)$, $r = r(t)$, $\theta = \theta(t)$, and $z = z(t)$. To calculate the (absolute) velocity vector $\mathbf{v}$ of the particle, we differentiate $\mathbf{r}(t)$:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} + \frac{dz}{dt}\mathbf{E}_z.$$

Now, using the chain rule, $\dot{\mathbf{e}}_r = \dot{\theta}\, d\mathbf{e}_r/d\theta = \dot{\theta}\mathbf{e}_\theta$. Also,

$$\frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r, \qquad \frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta.$$

[1] Details of these results are discussed in Section A.4 of Appendix A.

It follows that

$$\begin{aligned}\mathbf{v} &= \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta + \frac{dz}{dt}\mathbf{E}_z \\ &= \frac{dx}{dt}\mathbf{E}_x + \frac{dy}{dt}\mathbf{E}_y + \frac{dz}{dt}\mathbf{E}_z.\end{aligned}$$

To calculate the (absolute) acceleration vector $\mathbf{a}$, we differentiate $\mathbf{v}$ with respect to time:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\mathbf{e}_r\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\mathbf{e}_\theta\right) + \frac{d^2z}{dt^2}\mathbf{E}_z.$$

Using the chain rule to determine the time derivatives of the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$, and after collecting terms in the expressions for $\mathbf{a}$, the final form of the results is obtained:

$$\begin{aligned}\mathbf{a} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta + \frac{d^2z}{dt^2}\mathbf{E}_z \\ &= \frac{d^2x}{dt^2}\mathbf{E}_x + \frac{d^2y}{dt^2}\mathbf{E}_y + \frac{d^2z}{dt^2}\mathbf{E}_z.\end{aligned}$$

We have also included the representations for the velocity and acceleration vectors in Cartesian coordinates to emphasize the fact that the values of these vectors do not depend on the coordinate system used.

2.2.1 Common Errors

In my experience, the most common error with using cylindrical polar coordinates is to write $\mathbf{r} = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{E}_z$. *This is not true.* Another mistake is to differentiate $\mathbf{e}_r$ and $\mathbf{e}_\theta$ incorrectly with respect to time. Last, but not least, many people presume that all of the results presented here apply when θ is taken to be positive in the clockwise direction. Alas, this is not the case.

2.3 Kinetics of a Particle

Consider a particle of mass m. Let $\mathbf{F}$ denote the resultant external force acting on the particle, and let $\mathbf{G} = m\mathbf{v}$ be the linear momentum of the particle. Euler's first law (which is also known as Newton's second law or the balance of linear momentum) postulates that

$$\mathbf{F} = \frac{d\mathbf{G}}{dt} = m\mathbf{a}.$$

With respect to a Cartesian basis $\mathbf{F} = m\mathbf{a}$ is equivalent to three scalar equations:

$$F_x = ma_x = m\ddot{x}, \qquad F_y = ma_y = m\ddot{y}, \qquad F_z = ma_z = m\ddot{z},$$

where $\mathbf{F} = F_x\mathbf{E}_x + F_y\mathbf{E}_y + F_z\mathbf{E}_z$ and $\mathbf{a} = a_x\mathbf{E}_x + a_y\mathbf{E}_y + a_z\mathbf{E}_z$.

With respect to a cylindrical polar coordinate system, the single vector equation $\mathbf{F} = m\mathbf{a}$ is equivalent to three scalar equations:

$$(\mathbf{F} = m\mathbf{a}) \cdot \mathbf{e}_r : F_r = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right),$$

$$(\mathbf{F} = m\mathbf{a}) \cdot \mathbf{e}_\theta : F_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right),$$

$$(\mathbf{F} = m\mathbf{a}) \cdot \mathbf{E}_z : F_z = m\frac{d^2z}{dt^2}.$$

Finally, we recall for emphasis the relations

$$\begin{aligned}
\mathbf{e}_r &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y, \\
\mathbf{E}_x &= \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta, \\
\mathbf{E}_y &= \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta.
\end{aligned}$$

You will use these relations countless times in an undergraduate engineering dynamics course.

2.4 The Planar Pendulum

The planar pendulum is a classical problem in mechanics. As shown in Figure 2.3, a particle of mass m is suspended from a fixed point O either by an inextensible massless string or rigid massless rod of length L. The particle is free to move on a plane $(z = 0)$, and during its motion a vertical gravitational force $-mg\mathbf{E}_y$ acts on the particle.

We ask the following questions: what are the equations governing the motion of the particle and what is the tension in the string or rod? The answers to these questions are used to construct the motion of the particle and the plot of tension as a function of time that were shown in Figure 2.1 at the beginning of this chapter.

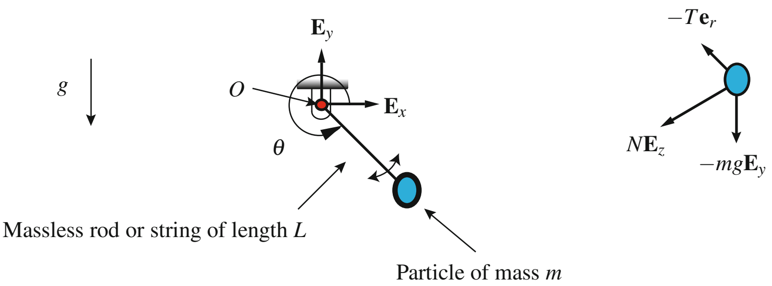

Fig. 2.3 *The planar pendulum and the free-body diagram of the particle of mass m.*

2.4.1 Kinematics

We begin by establishing some kinematical results. We note that $\mathbf{r} = L\mathbf{e}_r$. Differentiating with respect to t, and noting that L is constant, gives us the velocity $\mathbf{v}$. Similarly, we obtain $\mathbf{a}$ from $\mathbf{v}$:

$$\mathbf{v} = L\frac{d\mathbf{e}_r}{dt} = L\frac{d\theta}{dt}\mathbf{e}_\theta,$$
$$\mathbf{a} = L\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + L\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{dt} = L\frac{d^2\theta}{dt^2}\mathbf{e}_\theta - L\left(\frac{d\theta}{dt}\right)^2\mathbf{e}_r.$$

Alternatively, one can get these results by substituting $r = L$ and $z = 0$ in the general expressions recorded in Section 2.2. I do not recommend this approach inasmuch as it emphasizes memorization.

2.4.2 Forces

Next, as shown in Figure 2.3, we draw a free-body diagram. There is a tension force $-T\mathbf{e}_r$ and a normal force $N\mathbf{E}_z$ acting on a particle. The role of the tension force is to ensure that the distance of the particle from the origin is L and the normal force ensures that there is no motion in the direction of $\mathbf{E}_z$. These two forces are known as constraint forces. They are indeterminate (we need to use $\mathbf{F} = m\mathbf{a}$ to determine them). One should also note that the gravitational force has the representations

$$-mg\mathbf{E}_y = -mg\sin(\theta)\mathbf{e}_r - mg\cos(\theta)\mathbf{e}_\theta.$$

2.4.3 Balance Law

The third step is to write down the balance of linear momentum ($\mathbf{F} = m\mathbf{a}$):

$$-T\mathbf{e}_r + N\mathbf{E}_z - mg\mathbf{E}_y = mL\ddot{\theta}\mathbf{e}_\theta - mL\dot{\theta}^2\mathbf{e}_r.$$

We obtain three scalar equations from this vector equation:

$$mL\ddot{\theta} = -mg\cos(\theta), \qquad T = mL\dot{\theta}^2 - mg\sin(\theta), \qquad N = 0.$$

2.4.4 Analysis

The first of these equations is a second-order differential equation for $\theta(t)$:

$$mL\ddot{\theta} = -mg\cos(\theta).$$

Given the initial conditions $\theta(t_0)$ and $\dot{\theta}(t_0)$, one can solve this equation and determine the motion of the particle. Next, the second equation gives the tension T in the string or rod once $\theta(t)$ is known:

$$-T\mathbf{e}_r = -\left(mL\dot{\theta}^2 - mg\sin(\theta)\right)\mathbf{e}_r.$$

A representative example of the behavior of T during a motion of the pendulum is shown in Figure 2.1. This figure was constructed by first numerically solving the ordinary differential equations for $\theta(t)$ and then computing the corresponding $T(t)$.

For a string, it is normally assumed that $T > 0$, and for some motions of the string it is possible that this assumption is violated. In this case, the particle behaves as if it were free to move on the plane and $r \neq L$. Regardless, the normal force $N\mathbf{E}_z$ is zero in this problem.

2.5 Summary

In this chapter, the cylindrical polar coordinate system $\{r, \theta, z\}$ was introduced. To assist with certain expressions, the vectors $\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y$ and $\mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y$ were introduced. It was also shown that the position vector of a particle has the representations

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r + z\mathbf{E}_z = \sqrt{x^2+y^2}\mathbf{e}_r + z\mathbf{E}_z \\ &= r\cos(\theta)\mathbf{E}_x + r\sin(\theta)\mathbf{E}_y + z\mathbf{E}_z \\ &= x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z.\end{aligned}$$

By differentiating $\mathbf{r}$ with respect to time, the velocity and acceleration vectors were obtained. These vectors have the representations

$$\begin{aligned}\mathbf{v} = \frac{d\mathbf{r}}{dt} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{E}_z \\ &= \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y + \dot{z}\mathbf{E}_z, \\ \mathbf{a} = \frac{d\mathbf{v}}{dt} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{E}_z \\ &= \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y + \ddot{z}\mathbf{E}_z.\end{aligned}$$

To establish these results, the chain rule and the important identities $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$ were used.

Using a cylindrical polar coordinate system, $\mathbf{F} = m\mathbf{a}$ can be written as three scalar equations:

$$\begin{aligned}F_r &= m\left(\ddot{r} - r\dot{\theta}^2\right), \\ F_\theta &= m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right), \\ F_z &= m\ddot{z}.\end{aligned}$$

These equations were illustrated using the example of the planar pendulum.

2.6 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

2.1. Using Figure 2.2, verify that $\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y$ and $\mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y$. Then, by considering cases where $\mathbf{e}_r$ lies in the second, third, and fourth quadrants, verify that these definitions are valid for all values of θ.

2.2. Starting from the definitions $\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y$ and $\mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y$, show that $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$. In addition, verify that $\mathbf{E}_x = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta$ and $\mathbf{E}_y = \sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta$.

2.3. Calculate the velocity vectors of particles whose position vectors are $10\mathbf{e}_r$ and $5\mathbf{e}_r + t\mathbf{E}_z$, where $\theta = \pi t$. Why do all of these particles move with constant speed $\|\mathbf{v}\|$ yet have a nonzero acceleration?

2.4. The position vector of a particle of mass m that is placed at the end of a rotating telescoping rod is $\mathbf{r} = 6t\mathbf{e}_r$, where $\theta = 10t + 5$ (radians). Calculate the velocity and acceleration vectors of the particle, and determine the force $\mathbf{F}$ needed to sustain the motion of the particle. What is the force that the particle exerts on the telescoping rod?

2.5. In solving a problem, one person uses cylindrical polar coordinates whereas another uses Cartesian coordinates. To check that their answers are identical, they need to examine the relationship between the Cartesian and cylindrical polar components of a certain vector, say $\mathbf{b} = b_r\mathbf{e}_r + b_\theta\mathbf{e}_\theta$. To this end, show that

$$b_x = \mathbf{b}\cdot\mathbf{E}_x = b_r\cos(\theta) - b_\theta\sin(\theta), \qquad b_y = \mathbf{b}\cdot\mathbf{E}_y = b_r\sin(\theta) + b_\theta\cos(\theta).$$

2.6. Consider the projectile problem discussed in Section 1.6 of Chapter 1. Using a cylindrical polar coordinate system, show that the equations governing the motion of the particle are

$$m\ddot{r} - mr\dot{\theta}^2 = -mg\sin(\theta), \quad mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = -mg\cos(\theta), \quad m\ddot{z} = 0.$$

Notice that, in contrast to using Cartesian coordinates to determine the governing equations, solving these differential equations is nontrivial.

2.7. Consider a spherical bead of mass m and radius R that is placed inside a long cylindrical tube. The inner radius of the tube is R, and the tube is pivoted so that it rotates in a horizontal plane. Furthermore, the contact between the tube and the bead is smooth. Here, the bead is modeled as a particle of mass m. Now suppose that the tube is whirled at a constant angular speed Ω (radians per second). The whirling motion of the tube is such that the velocity vector of the bead is $\mathbf{v} = \dot{r}\mathbf{e}_r + \Omega r\mathbf{e}_\theta$. Show that the equation governing the motion of the bead is

$$\ddot{r} - \Omega^2 r = 0,$$

and the force exerted by the tube on the particle is $mg\mathbf{E}_z + 2m\dot{r}\Omega\mathbf{e}_\theta$.

2.8. Consider the case where the bead is initially at rest relative to the whirling tube at a location $r_0 = L$. Using the solution to the differential equation $\ddot{r} - \Omega^2 r = 0$ recorded in Section A.5.3 of Appendix A, show that, unless $L = 0$, the bead discussed in the previous exercise will eventually exit the whirling tube.

Chapter 3
Particles and Space Curves

TOPICS

In this chapter we discuss the differential geometry of space curves (a curve embedded in Euclidean three-space $\mathbb{E}^3$). In particular, we introduce the Serret-Frenet basis vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$. This is followed by the derivation of an elegant set of relations describing the rate of change of the tangent $\mathbf{e}_t$, principal normal $\mathbf{e}_n$, and binormal $\mathbf{e}_b$ vectors. Several examples of space curves are then discussed. We end the chapter with some applications to the mechanics of particles. Subsequent chapters also discuss several examples.

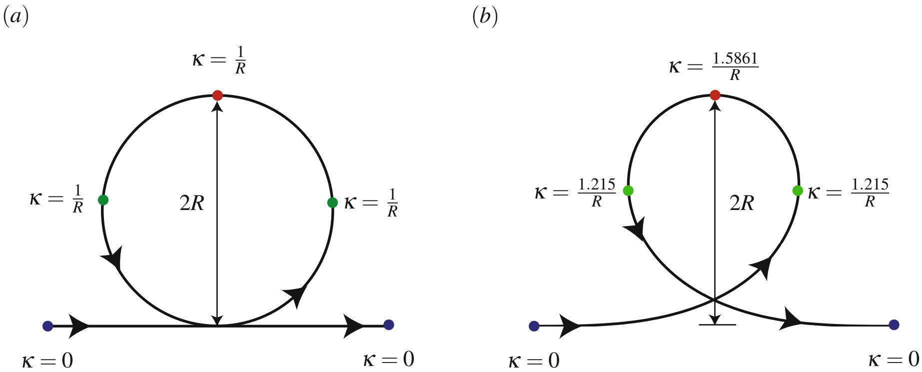

Fig. 3.1 *Two potential designs for a loop-the-loop track section of a roller coaster and selected values of the curvature κ along their lengths: (a) a design based on connecting two semicircular track sections of radius R with two straight track sections and (b) a design based on connecting two clothoid track sections.*

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3.1 Space Curves

For many systems, such as those featuring motion of a particle on a straight line or on a plane, it suffices to know how to manipulate Cartesian or polar coordinates. However, these coordinate systems become cumbersome for many problems. In general, the path of the particle is a curve in space and we refer to such curves as *space curves*. In many instances, this space curve is prescribed, as in the case of a particle moving on a circular path, or else it may lie on a surface. More often than not, this curve is not known a priori. In this chapter, the tools needed to analyze problems where a particle is moving on a space curve are developed and applied.

One of the most interesting applications of the tools we develop in this chapter is the design of roller coasters. A naive approach to designing a loop-the-loop section of a roller coaster would be to connect two circular arcs together (cf. Figure 3.1). Unfortunately, this design will lead to undesirable changes in acceleration at the base of the loop. Instead, dating to the mid-1970s, a design based on a curve known as a clothoid is used [60].[1] One of the differences in the two designs can be seen from a parameter known as the curvature κ which we shortly define. For the design featuring the circular arcs, κ is piecewise constant and κ varies in a linear manner for the clothoid design. In fact, moving along the clothoid loop shown in Figure 3.1(b), we would find that κ increases linearly with the distance traveled on the path from 0 to a maximum of $1.5861/R$ and then decreases linearly to zero.[2] In contrast for a circle-based design with the same total height of $2R$, the curvature would be 0 on the straight sections and jump instantaneously to $1/R$ on the circular sections of the path.

3.1.1 The Arc-Length Parameter

Consider a fixed curve $\mathscr{C}$ that is embedded in $\mathbb{E}^3$ (i.e., $\mathscr{C}$ is a space curve). Let the position vector of a point $P \in \mathscr{C}$ be denoted by $\mathbf{r}$. This vector has the representation

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z,$$

where x, y, and z are the usual Cartesian coordinates and $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ are the orthonormal basis vectors associated with these coordinates.

Associated with the curve $\mathscr{C}$, we can define an arc-length parameter s, where by definition,

$$\left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\frac{dx}{dt} + \frac{dy}{dt}\frac{dy}{dt} + \frac{dz}{dt}\frac{dz}{dt}.$$

[1] The clothoid loop is also used in the design of freeway exit ramps [60].

[2] We explore this matter in further detail later on in Section 5.8.4.

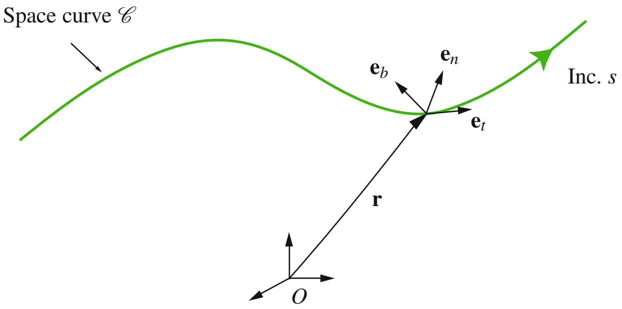

Fig. 3.2 *A space curve and the Serret-Frenet triad at one of its points.*

This parameter uniquely identifies a point P of $\mathscr{C}$, and we can use it to obtain a different representation of the position vector of a point on the curve:

$$\mathbf{r} = \hat{\mathbf{r}}(s).$$

3.1.2 The Serret-Frenet Triad

In the sequel we need to examine the forces acting on a particle as it moves on a curve and use this information to determine the motion of the particle. For cases where the particle is moving on a straight line or a circular path, we can use Cartesian and cylindrical polar coordinate systems, respectively, to simplify our analysis. However when the particle is moving on a curve in the form of a parabola or clothoid, it is not convenient to use these two coordinate systems. We now turn to examining a set of basis vectors that will help us for these more complex cases.

Consider a fixed space curve such as the one shown in Figure 3.2. We wish to define the so-called Serret-Frenet basis vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ for a point P of this space curve.[3]

To define a tangent vector, we consider two points P and P' of $\mathscr{C}$, where the position vectors of P and P' are $\hat{\mathbf{r}}(s)$ and $\hat{\mathbf{r}}(s + \Delta s)$, respectively. We define the vector $\mathbf{e}_t$ as

$$\mathbf{e}_t = \hat{\mathbf{e}}_t(s) = \frac{d\mathbf{r}}{ds} = \lim_{\triangle s \to 0} \frac{\hat{\mathbf{r}}(s + \Delta s) - \hat{\mathbf{r}}(s)}{\Delta s}.$$

[3] These triads and formulae for their rates of change were established by Jean-Frédéric Frenet (1816–1900) in 1847 (see [28]) and Joseph Alfred Serret (1819–1885) in 1851 (see [68]). One can extend our forthcoming discussion to curves that are moving in space, for example, a curve in the form of a circle whose radius changes with time. In this case $\mathbf{r} = \mathbf{r}(s,t)$, and the Serret-Frenet triad is obtained by the partial differentiation of $\mathbf{r}$ with respect to s. Such an extension is well known and needed for discussing the theory of rods but is beyond the scope of an undergraduate engineering dynamics course.

Because $\hat{\mathbf{r}}(s+\Delta s)-\hat{\mathbf{r}}(s) \to \Delta s \mathbf{e}_t$ as $\Delta s \to 0$,[4] $\mathbf{e}_t$ is known as the unit tangent vector to $\mathscr{C}$ at the point P (see Figure 3.2).

We next consider the second derivative of $\mathbf{r}$ with respect to s:

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{e}_t}{ds}.$$

After noting that $\mathbf{e}_t \cdot \mathbf{e}_t = 1$ and then differentiating this relation, we find that $d\mathbf{e}_t/ds$ is perpendicular to $\mathbf{e}_t$. We define κ to be the magnitude of $d\mathbf{e}_t/ds$ and $\mathbf{e}_n$ to be its direction:

$$\kappa \mathbf{e}_n = \frac{d\mathbf{e}_t}{ds}.$$

The vector $\mathbf{e}_n$ is known as the (unit) principal normal vector to $\mathscr{C}$ at the point P, and the scalar κ is known as the curvature of $\mathscr{C}$ at the point P.[5] It should be noted that $\mathbf{e}_n$ and κ are functions of s:

$$\kappa = \hat{\kappa}(s), \qquad \mathbf{e}_n = \hat{\mathbf{e}}_n(s).$$

Often an additional variable ρ, which is known as the radius of curvature, is defined:

$$\rho = \hat{\rho}(s) = \frac{1}{\kappa}.$$

For the degenerate case where $d\mathbf{e}_t/ds = \mathbf{0}$ for a particular s, the curvature κ is 0 and the vector $\mathbf{e}_n$ is not uniquely defined. In this case, one usually defines $\mathbf{e}_n$ to be a unit vector perpendicular to $\mathbf{e}_t$. The most common case of this occurrence is when the curve $\mathscr{C}$ is a straight line.

We show that a circle of radius R has a curvature of $1/R$. Another special plane curve of particular interest is the clothoid shown in Figure 3.3. For this curve, κ is a linear function of s.[6] It is easy to see how one can use the clothoid shown in Figure 3.3 to smoothly connect a straight line to an arc of a circle of radius $a/2\pi$.

The final vector of interest is $\mathbf{e}_b$, and it is defined by

$$\mathbf{e}_b = \hat{\mathbf{e}}_b(s) = \mathbf{e}_t \times \mathbf{e}_n.$$

Clearly, $\mathbf{e}_b$ is a unit vector. It is known as the (unit) binormal vector to $\mathscr{C}$ at the point P.

It should be noted that the set of vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ is defined for each point of $\mathscr{C}$, is orthonormal, and, because $\mathbf{e}_b \cdot (\mathbf{e}_t \times \mathbf{e}_n) = 1$, forms a right-handed set. We refer to the set $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ as the Serret-Frenet triad. Because this set is orthonormal it may

[4] This may be easily seen by sketching the curve and the position vectors of P and P' and then taking the limit as Δs tends to 0.

[5] Often the convention that κ is nonnegative is adopted. This allows the convenient identification that $\mathbf{e}_n$ points in the direction of $d\mathbf{e}_t/ds$. We adhere strictly to this convention.

[6] This result is left as an exercise for the interested reader (see Exercise 3.11 on page 51).

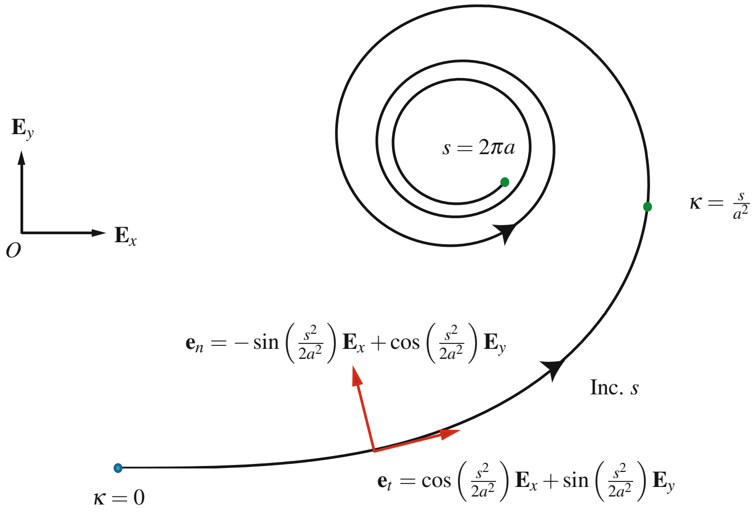

Fig. 3.3 *A section of a plane curve known as a clothoid. For this curve,* $\kappa = s/a^2$ *where a is a constant.*

be used as a basis for $\mathbb{E}^3$. That is, given any vector $\mathbf{b}$, one has the representations

$$\mathbf{b} = b_t\mathbf{e}_t + b_n\mathbf{e}_n + b_b\mathbf{e}_b = b_x\mathbf{E}_x + b_y\mathbf{E}_y + b_z\mathbf{E}_z,$$

where $b_t = \mathbf{b}\cdot\mathbf{e}_t$, and so on.

At a particular s, the plane defined by the vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ is known as the *osculating plane*, and the plane defined by the vectors $\mathbf{e}_t$ and $\mathbf{e}_b$ is known as the *rectifying plane*. These planes will, in general, depend on the particular point P of $\mathscr{C}$.[7]

3.2 The Serret-Frenet Formulae

These three formulae relate the rate of change of the vectors $\mathbf{e}_t$, $\mathbf{e}_n$, and $\mathbf{e}_b$ with respect to the arc-length parameter s to the set of vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$.

Consider the vector $\mathbf{e}_t$. Recalling one of the previous results, the first of the desired formulae is recorded:

$$\frac{d\mathbf{e}_t}{ds} = \kappa\mathbf{e}_n.$$

[7] It is beyond our purposes to present an additional discussion on these planes. The interested reader is referred to an introductory text on differential geometry, several of which are available. We mention in particular Kreyszig [42], Spivak [73], and Struik [77].

It is convenient to consider next the vector $\mathbf{e}_b$. This vector is a unit vector, therefore it cannot change along its own length. Mathematically, we see this by noting that

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n \Longrightarrow \mathbf{e}_b \cdot \mathbf{e}_b = 1$$
$$\Longrightarrow \frac{d\mathbf{e}_b}{ds} \cdot \mathbf{e}_b = 0.$$

Consequently, $d\mathbf{e}_b/ds$ has components only in the directions of $\mathbf{e}_t$ and $\mathbf{e}_n$. Let's first examine what the component of $d\mathbf{e}_b/ds$ is in the $\mathbf{e}_t$ direction:

$$\mathbf{e}_t \cdot \mathbf{e}_b = 0 \Longrightarrow \frac{d\mathbf{e}_t}{ds} \cdot \mathbf{e}_b + \frac{d\mathbf{e}_b}{ds} \cdot \mathbf{e}_t = 0$$
$$\Longrightarrow \kappa \mathbf{e}_n \cdot \mathbf{e}_b + \frac{d\mathbf{e}_b}{ds} \cdot \mathbf{e}_t = 0$$
$$\Longrightarrow \frac{d\mathbf{e}_b}{ds} \cdot \mathbf{e}_t = 0.$$

It now follows that $d\mathbf{e}_b/ds$ is parallel to $\mathbf{e}_n$. Consequently, we define

$$\frac{d\mathbf{e}_b}{ds} = -\tau \mathbf{e}_n,$$

where $\tau = \hat{\tau}(s)$ is the torsion of the curve $\mathscr{C}$ at the particular point P corresponding to the value of s. The negative sign in the above formula is conventional.

To obtain the final Serret-Frenet formula for $d\mathbf{e}_n/ds$, we perform a direct calculation:

$$\frac{d\mathbf{e}_n}{ds} = \frac{d}{ds}(\mathbf{e}_b \times \mathbf{e}_t) = \frac{d\mathbf{e}_b}{ds} \times \mathbf{e}_t + \mathbf{e}_b \times \frac{d\mathbf{e}_t}{ds}$$
$$= (-\tau \mathbf{e}_n) \times \mathbf{e}_t + \mathbf{e}_b \times (\kappa \mathbf{e}_n).$$

After simplifying this result by evaluating the cross products, we obtain

$$\frac{d\mathbf{e}_n}{ds} = -\kappa \mathbf{e}_t + \tau \mathbf{e}_b.$$

The Serret-Frenet formulae can be conveniently summarized as

$$\begin{bmatrix} \frac{d\mathbf{e}_t}{ds} \\ \frac{d\mathbf{e}_n}{ds} \\ \frac{d\mathbf{e}_b}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_t \\ \mathbf{e}_n \\ \mathbf{e}_b \end{bmatrix}.$$

One can define what is often referred to as the Darboux[8] vector $\boldsymbol{\omega}_{\text{SF}}$:

$$\boldsymbol{\omega}_{\text{SF}} = \kappa \mathbf{e}_b + \tau \mathbf{e}_t.$$

[8] Gaston Darboux (1842–1917) was a French mathematician who wrote an authoritative four-volume treatise on differential geometry, which was published between 1887 and 1896 (see [21]).

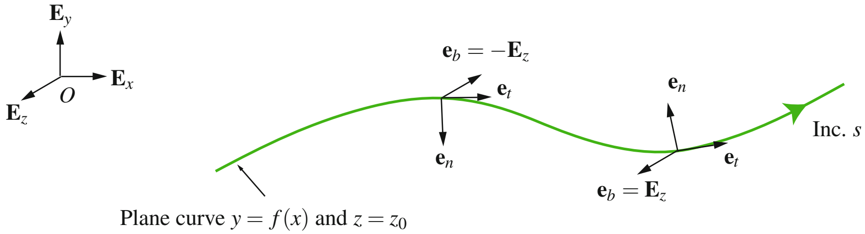

Fig. 3.4 *An example of a plane curve and the Serret-Frenet triad* $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ *at two distinct points along its length.*

Using the Darboux vector, the Serret-Frenet relations can also be written in the form

$$\frac{d\mathbf{e}_i}{ds} = \boldsymbol{\omega}_{\mathrm{SF}} \times \mathbf{e}_i,$$

where $i = t$, n, or b.

3.3 Examples of Space Curves

We proceed to discuss four examples of spaces curves: a plane curve, a circle, a space curve parametrized by x, and a circular helix. The degenerate case of a straight line is discussed in conjunction with the plane curve.

3.3.1 A Curve on a Plane

As shown in Figure 3.4,[9] consider a curve on a plane in $\mathbb{E}^3$. The plane is defined by the relation $z = z_0$, and the curve is defined by the intersection of two 2-dimensional surfaces:

$$z = z_0, \qquad y = f(x),$$

where we assume that f is as smooth as necessary. A specific example is presented in Section 3.5.

The position vector of a point P on this curve is

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z = x\mathbf{E}_x + f(x)\mathbf{E}_y + z_0\mathbf{E}_z.$$

To determine the arc-length parameter s of the curve, we first note that

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{E}_x + \frac{df}{dx}\frac{dx}{dt}\mathbf{E}_y.$$

[9] The convention that $\kappa \geq 0$ has been used to define the vector $\mathbf{e}_n$.

Consequently,

$$\left(\frac{ds}{dt}\right)^2 = \left(1+\left(\frac{df}{dx}\right)^2\right)\left(\frac{dx}{dt}\right)^2,$$

or, assuming that s increases in the direction of increasing x,

$$\frac{ds}{dt} = \sqrt{\left(1+\left(\frac{df}{dx}\right)^2\right)}\frac{dx}{dt}.$$

Integrating both sides of the above equation, we obtain[10]

$$s = s(x) = \int_{x_0}^{x}\sqrt{1+\left(\frac{df}{dx}\right)^2}\,du + s(x_0).$$

As follows from our previous development, we should invert $s(x)$ to determine $x(s)$. However, because we prefer to keep the function $f(x)$ arbitrary, we express the results as functions of x.

To determine the tangent vector, we recall its definition and use the chain rule:

$$\mathbf{e}_t = \mathbf{e}_t(x) = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dx}\frac{dx}{ds} = \frac{1}{\sqrt{1+\left(\frac{df}{dx}\right)^2}}\left(\mathbf{E}_x + \frac{df}{dx}\mathbf{E}_y\right),$$

where we have also used the identity

$$\frac{dx}{ds} = \left(\frac{ds}{dx}\right)^{-1}.$$

The expected unit magnitude of $\mathbf{e}_t$ should be noted.

The principal normal vector $\mathbf{e}_n$ and the curvature κ are determined by evaluating the derivative of $\mathbf{e}_t$ with respect to s:

$$\kappa\mathbf{e}_n = \frac{d\mathbf{e}_t}{ds} = \frac{d\mathbf{e}_t}{dx}\frac{dx}{ds} = \frac{d\mathbf{e}_t}{dx}\left(\frac{ds}{dx}\right)^{-1}.$$

Omitting the details of the calculation, after some subsequent rearrangement we obtain

$$\kappa\mathbf{e}_n = \frac{\frac{d^2f}{dx^2}}{\left(1+\left(\frac{df}{dx}\right)^2\right)^2}\left(\mathbf{E}_y - \frac{df}{dx}\mathbf{E}_x\right).$$

---

[10] It is tacitly understood that we need to express df/dx in terms of the dummy variable u in order to evaluate this integral.

Recalling that $\mathbf{e}_n$ is a unit vector and that κ is positive, the final results are obtained:

$$\kappa = \kappa(x) = \frac{\left|\frac{d^2f}{dx^2}\right|}{\left(\sqrt{1+\left(\frac{df}{dx}\right)^2}\right)^3},$$

$$\mathbf{e}_n = \mathbf{e}_n(x) = \frac{\operatorname{sgn}\left(\frac{d^2f}{dx^2}\right)}{\sqrt{1+\left(\frac{df}{dx}\right)^2}}\left(\mathbf{E}_y - \frac{df}{dx}\mathbf{E}_x\right).$$

Here, $\operatorname{sgn}(a) = 1$ if $a > 0$ and -1 if $a < 0$.

Finally, the binormal vector $\mathbf{e}_b$ may be determined:

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = \operatorname{sgn}\left(\frac{d^2f}{dx^2}\right)\mathbf{E}_z,$$

and, because this vector is a piecewise constant, the torsion of the curve is

$$\tau = 0.$$

Returning briefly to Figure 3.4, you may have noticed that for certain segments of the curve $\mathbf{e}_b = \mathbf{E}_z$ and for others $\mathbf{e}_b = -\mathbf{E}_z$. The points where this transition occurs are those where $d^2f/dx^2 = 0$. At these points, $\kappa = 0$ and $\mathbf{e}_n$ is not defined by the Serret-Frenet formula $d\mathbf{e}_t/ds = \kappa\mathbf{e}_n$.

For the plane curve, many texts use a particular representation of the tangent and normal vectors by defining an angle $\beta = \beta(s)$:

$$\mathbf{e}_t = \cos(\beta(s))\,\mathbf{E}_x + \sin(\beta(s))\,\mathbf{E}_y,$$

$$\mathbf{e}_n = \cos(\beta(s))\,\mathbf{E}_y - \sin(\beta(s))\,\mathbf{E}_x.$$

Notice that $\mathbf{e}_t$ and $\mathbf{e}_n$ are unit vectors, as expected. By differentiating these expressions with respect to s, one finds that

$$\kappa = \frac{d\beta}{ds}.$$

Consequently, κ can be interpreted as a rate of rotation of the vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ about $\mathbf{e}_b = \pm\mathbf{E}_z$.

3.3.1.1 The Straight Line

A special case of the plane curve arises when $f(x) = ax + b$, where a and b are constants. In this case, we find from above that

$$\mathbf{e}_t = \mathbf{e}_t(x) = \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2}} \left(\mathbf{E}_x + \frac{df}{dx}\mathbf{E}_y\right) = \frac{1}{\sqrt{1 + a^2}} \left(\mathbf{E}_x + a\mathbf{E}_y\right).$$

It should be clear that we are assuming that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{df}{dx}\right)^2},$$

as opposed to $-\sqrt{1 + (df/dx)^2}$. Turning to the principal normal vector, because $d\mathbf{e}_t/ds = \mathbf{0}$, the curvature $\kappa = 0$ and $\mathbf{e}_n$ is not defined. For consistency, it is convenient to choose $\mathbf{e}_n$ to be perpendicular to $\mathbf{e}_t$. The binormal vector is then defined by $\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n$.

3.3.2 A Space Curve Parametrized by x

Consider a curve $\mathscr{C}$ in $\mathbb{E}^3$. Suppose that the curve is defined by the intersection of the two 2-dimensional surfaces

$$z = g(x), \qquad y = f(x),$$

where we assume that f and g are as smooth as necessary. The plane curve discussed in Section 3.3.1 can be considered as a particular example of a space curve. The position vector of a point P on this curve is

$$\mathbf{r} = x\mathbf{E}_x + f(x)\mathbf{E}_y + g(x)\mathbf{E}_z.$$

The arc-length parameter s may be determined in a manner similar to what was discussed previously:

$$s = s(x) = \int_{x_0}^{x} \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{dg}{dx}\right)^2} du + s(x_0).$$

As in Section 3.3.1, we assume that the function $x(s)$ is available to us and express all of our results as functions of x. It follows that we can later express all of our results as functions of s if required.

To calculate the tangent vector, we use the chain rule as before:

$$\begin{aligned}\mathbf{e}_t &= \mathbf{e}_t(x) = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dx}\frac{dx}{ds} \\ &= \frac{1}{\sqrt{1+\left(\frac{df}{dx}\right)^2+\left(\frac{dg}{dx}\right)^2}}\left(\mathbf{E}_x + \frac{df}{dx}\mathbf{E}_y + \frac{dg}{dx}\mathbf{E}_z\right).\end{aligned}$$

The principal normal vector $\mathbf{e}_n$ and the curvature κ are determined by evaluating the derivative of $\mathbf{e}_t$ with respect to s:

$$\kappa\mathbf{e}_n = \frac{d\mathbf{e}_t}{ds} = \frac{d\mathbf{e}_t}{dx}\frac{dx}{ds} = \frac{d\mathbf{e}_t}{dx}\left(\frac{ds}{dx}\right)^{-1}.$$

Finally, the binormal vector $\mathbf{e}_b$ may be determined using its definition. We omit details of the expressions for the principal normal $\mathbf{e}_n$ and binormal $\mathbf{e}_b$ vectors, curvature κ, and the torsion τ. They may be obtained using the quoted relations, and their specific general forms are not of further interest here.

3.3.3 A Circle on a Plane

As shown in Figure 3.5, consider a curve in the form of a circle that lies on a plane in $\mathbb{E}^3$. The plane is defined by the relation $z = z_0$, and the curve is defined by the intersection of two 2-dimensional surfaces:

$$z = z_0, \qquad r = R = \sqrt{x^2+y^2}.$$

In addition, it is convenient to recall the relations

$$\theta = \tan^{-1}\left(\frac{y}{x}\right),$$

$$\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_\theta = \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x.$$

The position vector of a point P on this curve is

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z_0\mathbf{E}_z = R\mathbf{e}_r + z_0\mathbf{E}_z.$$

To determine the arc-length parameter s of the curve, we first note that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R\frac{d\mathbf{e}_r}{dt} = R\frac{d\theta}{dt}\mathbf{e}_\theta.$$

Consequently,

$$\frac{ds}{dt} = R\frac{d\theta}{dt}.$$

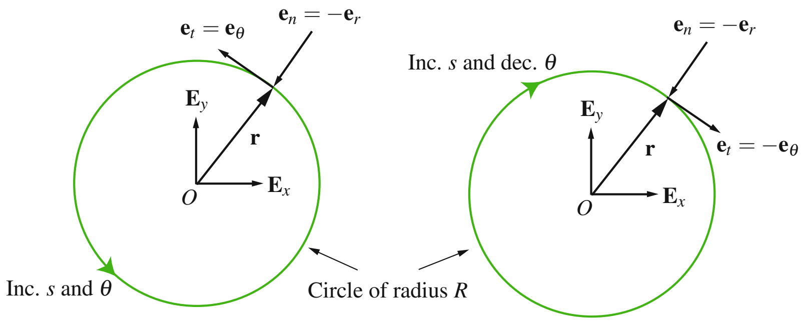

Fig. 3.5 *A circle of radius R. On the left, the direction of increasing s and θ are identical, whereas they are opposite to each other in the right-hand side image.*

Here, we have assumed that s increases in the direction of increasing θ.[11] Integrating both sides of this equation, we obtain

$$s(\theta) = R(\theta - \theta_0) + s(\theta_0).$$

Fortunately, we can invert the function $s(\theta)$ to solve for $\theta(s)$:

$$\theta(s) = \frac{1}{R}(s - s_0) + \theta(s_0).$$

The previous result enables us to write

$$\mathbf{r} = \hat{\mathbf{r}}(s) = R\hat{\mathbf{e}}_r(s) + z_0\mathbf{E}_z,$$

where

$$\hat{\mathbf{e}}_r(s) = \cos(\theta(s))\mathbf{E}_x + \sin(\theta(s))\mathbf{E}_y.$$

It should be noted that the function $\hat{\mathbf{e}}_\theta(s)$ can be defined in a similar manner.

To determine the tangent vector, we differentiate $\mathbf{r}$ as a function of s:

$$\mathbf{e}_t = \hat{\mathbf{e}}_t(s) = \frac{d\mathbf{r}}{ds} = R\frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{ds} = R\mathbf{e}_\theta\frac{1}{R} = \mathbf{e}_\theta.$$

The expected unit magnitude of $\mathbf{e}_t$ should again be noted. The principal normal vector $\mathbf{e}_n$ and the curvature κ are determined by evaluating the derivative of $\mathbf{e}_t$ with respect to s:

$$\kappa\mathbf{e}_n = \hat{\kappa}(s)\hat{\mathbf{e}}_n(s) = \frac{d\mathbf{e}_t}{ds} = \frac{d\mathbf{e}_\theta}{d\theta}\frac{d\theta}{ds} = -\frac{1}{R}\mathbf{e}_r.$$

[11] Otherwise, $ds/d\theta = -R$ and the forthcoming results need some minor modifications. As can be seen from Figure 3.5, when $ds/d\theta = -R$, $\mathbf{e}_t = -\mathbf{e}_\theta$, $\mathbf{e}_n = -\mathbf{e}_r$, and $\mathbf{e}_b = -\mathbf{E}_z$.

Recalling that $\mathbf{e}_n$ has unit magnitude and adopting the convention that $\kappa \geq 0$, we obtain

$$\kappa = \frac{1}{R}, \qquad \mathbf{e}_n = -\mathbf{e}_r.$$

Finally, the binormal vector $\mathbf{e}_b$ may be determined:

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = \mathbf{E}_z.$$

Because this vector is constant, the torsion of the circular curve is, trivially,

$$\tau = 0.$$

In summary, one has for a circle in the $\mathbf{E}_x - \mathbf{E}_y$ plane,

$$\mathbf{e}_t = \mathbf{e}_\theta, \qquad \mathbf{e}_n = -\mathbf{e}_r, \qquad \mathbf{e}_b = \mathbf{E}_z, \qquad \rho = R, \qquad \tau = 0.$$

Based on previous work with cylindrical polar coordinates, the tangent and normal vectors for this curve should have been anticipated.

3.3.4 A Circular Helix

Consider a curve in the form of a helix that is embedded in $\mathbb{E}^3$.[12] The helix is defined by the intersection of a pair of two-dimensional surfaces. One of these surfaces is a cylinder defined by the relation

$$r = R,$$

where $\{r, \theta, z\}$ are the usual cylindrical polar coordinates:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

The second surface is defined by the equation

$$z = g(r, \theta) = \alpha r \theta.$$

This surface is known as a helicoid.[13] Taking the intersection of the cylinder and helicoid one obtains a circular helix. The helix is right-handed if $\alpha > 0$, left-handed if $\alpha < 0$, and degenerates to a circle if $\alpha = 0$. An example of a right-handed circular helix is shown in Figure 3.6. The angle γ whose tangent is α is known as the pitch angle of the helix.

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[12] This is an advanced example. According to Kreyszig [42], the circular helix is the only nontrivial example of a curve with constant torsion and constant curvature.

[13] See, for example, Section 2.2 and Fig. 3-4 of Struik [77].

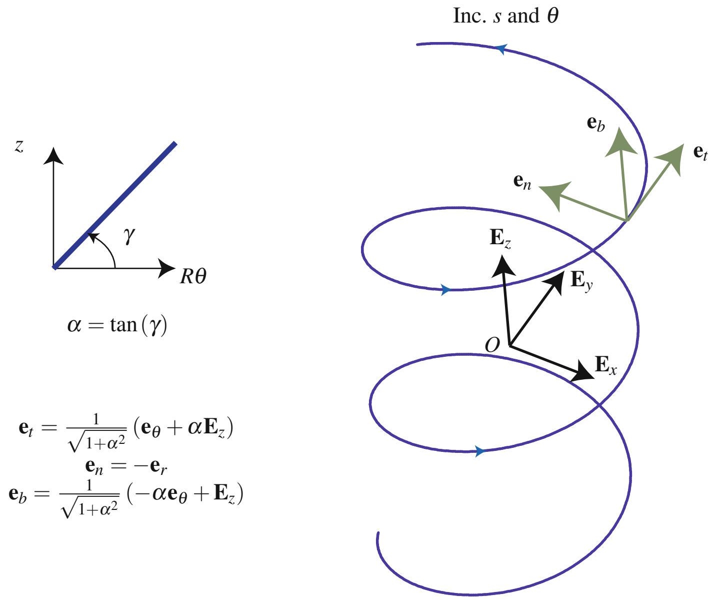

Fig. 3.6 *Examples of Serret-Frenet triads for a portion of a (right-handed) circular helix. The inset image shows the relationship between the pitch angle γ and the parameter α.*

In Cartesian coordinates, the circular helix may be represented by

$$x = R\cos(\theta), \qquad y = R\sin(\theta), \qquad z = \alpha R\theta.$$

The position vector of a point P on the helix is

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z = R\mathbf{e}_r + \alpha R\theta\mathbf{E}_z,$$

where for convenience we have defined, as always,

$$\mathbf{e}_r = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_\theta = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y.$$

To determine the arc-length parameter s of the curve we first note that

$$\frac{d\mathbf{r}}{dt} = R\frac{d\mathbf{e}_r}{dt} + \frac{d}{dt}(\alpha R\theta\mathbf{E}_z) = R\frac{d\theta}{dt}\mathbf{e}_\theta + \alpha R\frac{d\theta}{dt}\mathbf{E}_z.$$

Consequently,

$$\frac{ds}{dt} = R\sqrt{1+\alpha^2}\frac{d\theta}{dt}.$$

Here, we have assumed that s increases in the direction of increasing θ.[14] Integrating both sides of this equation we obtain

$$s(\theta) = R\sqrt{1+\alpha^2}(\theta - \theta_0) + s(\theta_0).$$

Fortunately, as in the case of the plane circle, we can invert the function $s(\theta)$ to solve for $\theta(s)$:

$$\theta(s) = \frac{1}{R\sqrt{1+\alpha^2}}(s - s_0) + \theta(s_0).$$

The previous result enables us to write $\mathbf{r}$ as a function of s:

$$\mathbf{r} = \hat{\mathbf{r}}(s) = R\hat{\mathbf{e}}_r(s) + \left(\frac{1}{R\sqrt{1+\alpha^2}}(s - s_0) + \theta(s_0)\right) R\alpha\mathbf{E}_z,$$

where

$$\hat{\mathbf{e}}_r(s) = \cos(\theta(s))\,\mathbf{E}_x + \sin(\theta(s))\,\mathbf{E}_y.$$

It should be noted that the function $\hat{\mathbf{e}}_\theta(s)$ can be defined in a similar manner.

To determine the tangent vector, we differentiate $\mathbf{r}$ with respect to s:

$$\mathbf{e}_t = \hat{\mathbf{e}}_t(s) = \frac{d\mathbf{r}}{ds} = R\frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{ds} + R\alpha\frac{d\theta}{ds}\mathbf{E}_z = \frac{1}{\sqrt{1+\alpha^2}}(\mathbf{e}_\theta + \alpha\mathbf{E}_z).$$

The expected unit magnitude of $\mathbf{e}_t$ should again be noted. The principal normal vector $\mathbf{e}_n$ and the curvature κ are determined as usual by evaluating the derivative of $\mathbf{e}_t$ with respect to s:

$$\begin{aligned}\kappa\mathbf{e}_n &= \hat{\kappa}(s)\hat{\mathbf{e}}_n(s) = \frac{d\mathbf{e}_t}{ds} = \frac{1}{\sqrt{1+\alpha^2}}\frac{d\mathbf{e}_\theta}{ds} + \frac{d}{ds}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\mathbf{E}_z\right)\\ &= -\frac{1}{R(1+\alpha^2)}\mathbf{e}_r.\end{aligned}$$

Recalling that the vector $\mathbf{e}_n$ has unit magnitude and adopting the convention that $\kappa \geq 0$, we obtain

$$\kappa = \frac{1}{R(1+\alpha^2)}, \qquad \mathbf{e}_n = -\mathbf{e}_r.$$

Finally, the binormal vector $\mathbf{e}_b$ may be determined:

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n = \frac{1}{\sqrt{1+\alpha^2}}(-\alpha\mathbf{e}_\theta + \mathbf{E}_z).$$

[14] Otherwise, $ds/d\theta = -R\sqrt{1+\alpha^2}$ and the forthcoming results need some minor modifications: $\mathbf{e}_n$ remains equal to $-\mathbf{e}_r$ but $\mathbf{e}_t \rightarrow -\mathbf{e}_t$ and $\mathbf{e}_b \rightarrow -\mathbf{e}_b$.

The binormal vector is not constant. Differentiating $\mathbf{e}_b$ with respect to s and rearranging the resulting expression reveals that the torsion of the circular helix is

$$\tau = \frac{\alpha}{R(1+\alpha^2)}.$$

Observe that the torsion is constant. Indeed, $\tau = \alpha\kappa$ for a circular helix.

3.4 Application to Particle Mechanics

In applications of the Serret-Frenet formulae to particle dynamics we make two identifications:

1. The space curve $\mathscr{C}$ is identified as the path of the particle.
2. The arc-length parameter s is considered to be a function of t.

In particular, we note that s may be identified as the distance traveled along the curve $\mathscr{C}$ from a given reference point.

With the identifications in mind, the position vector $\mathbf{r}$ of the particle can be given the equivalent functional representations

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z = \mathbf{r}(t) = \hat{\mathbf{r}}\left(s(t)\right).$$

Using these representations, we obtain for the velocity vector $\mathbf{v}$ of the particle

$$\mathbf{v} = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y + \dot{z}\mathbf{E}_z = \dot{\mathbf{r}}(t) = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\mathbf{e}_t.$$

It is important to see here that

$$\mathbf{v} = \frac{ds}{dt}\mathbf{e}_t = v\mathbf{e}_t.$$

Because the speed v is greater than or equal to zero, it is apparent that $\mathbf{e}_t$ is the unit vector in the direction of $\mathbf{v}$. Similarly, for the acceleration vector $\mathbf{a}$ we obtain the expression

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{e}_t\right) = \frac{d^2s}{dt^2}\mathbf{e}_t + \frac{ds}{dt}\frac{d\mathbf{e}_t}{dt} = \frac{d^2s}{dt^2}\mathbf{e}_t + \frac{ds}{dt}\frac{d\mathbf{e}_t}{ds}\frac{ds}{dt}.$$

Recalling the definitions of the principal normal vector $\mathbf{e}_n$ and the speed $v = \dot{s}$, we obtain the final desired expression for $\mathbf{a}$:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d^2s}{dt^2}\mathbf{e}_t + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{e}_n = \frac{dv}{dt}\mathbf{e}_t + \kappa v^2\mathbf{e}_n.$$

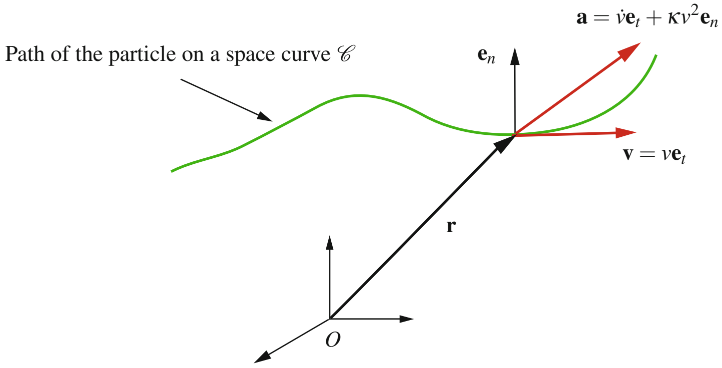

Fig. 3.7 *The velocity* **v** *and acceleration* **a** *vectors of a particle moving on a space curve.*

This remarkable result states that the acceleration vector of the particle lies entirely in the osculating plane (see Figure 3.7).

We note in passing that the distance traveled by the particle along its path may be determined from the vector **v**. To see this, we first recall that

$$\mathbf{v} \cdot \mathbf{v} = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = \left(\frac{ds}{dt}\right)^2 .$$

This implies that

$$s - s_0 = \int_{s_0}^{s} ds = \int_{t_0}^{t} \sqrt{\mathbf{v}(\mu) \cdot \mathbf{v}(\mu)}\, d\mu,$$

where s_0 denotes the value of s when $t = t_0$.

For a particle of mass m, Newton's second law states that

$$\mathbf{F} = m\mathbf{a},$$

where **F** is the resultant external force acting on the particle. Recalling that, for each s, the set of vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ forms a basis for $\mathbb{E}^3$, we may write

$$\mathbf{F} = F_t\mathbf{e}_t + F_n\mathbf{e}_n + F_b\mathbf{e}_b,$$

where

$$\begin{aligned}
F_t &= \hat{F}_t(s) = \mathbf{F} \cdot \hat{\mathbf{e}}_t(s), \\
F_n &= \hat{F}_n(s) = \mathbf{F} \cdot \hat{\mathbf{e}}_n(s), \\
F_b &= \hat{F}_b(s) = \mathbf{F} \cdot \hat{\mathbf{e}}_b(s).
\end{aligned}$$

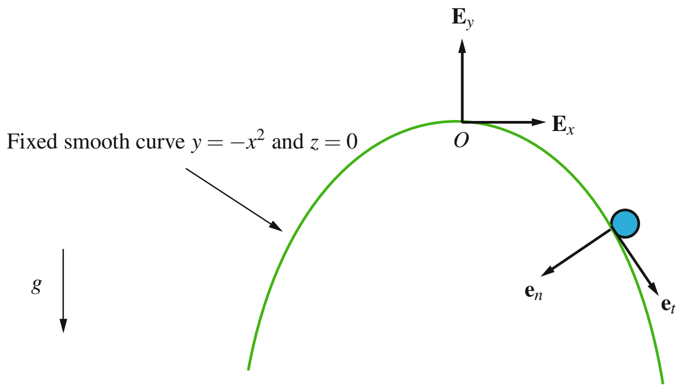

Fig. 3.8 *A particle moving on a curve under gravity.*

Using these results, we find that

$$(\mathbf{F} = m\mathbf{a}) \cdot \mathbf{e}_t : F_t = m\frac{d^2s}{dt^2},$$
$$(\mathbf{F} = m\mathbf{a}) \cdot \mathbf{e}_n : F_n = m\kappa\left(\frac{ds}{dt}\right)^2,$$
$$(\mathbf{F} = m\mathbf{a}) \cdot \mathbf{e}_b : F_b = 0.$$

In certain cases, these three equations are completely uncoupled and allow a problem in particle dynamics to be easily solved. We note that the previous equations also imply that $\mathbf{F}$ lies entirely in the osculating plane.

3.5 A Particle Moving on a Fixed Curve Under Gravity

As shown in Figure 3.8, we consider a particle of mass m moving on a smooth plane curve defined by $y = f(x) = -x^2$ and $z = 0$. A vertical gravitational force $-mg\mathbf{E}_y$ acts on the particle. We seek to determine the differential equation governing the motion of the particle and the force exerted by the curve on the particle. In addition, we examine the conditions on the motion of the particle that result in its losing contact with the curve.

3.5.1 Kinematics

We first consider the Serret-Frenet triad for this plane curve. From the results in Section 3.1.2, we find that the arc-length parameter s of the curve as a function of x

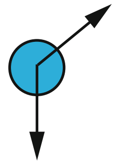

Fig. 3.9 *Free-body diagram of the particle.*

is

$$s = s(x) = \int_0^x \sqrt{1+4u^2}du + s(0)$$
$$= \frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\sinh^{-1}(2x) + s(0).$$

Here, we have taken the arbitrary constant $x_0 = 0$. One can also determine the radius of curvature:

$$\rho = \frac{1}{2}\left(\sqrt{1+4x^2}\right)^3.$$

The Serret-Frenet triad as a function of x can be calculated:

$$\begin{aligned}
\mathbf{e}_t &= \frac{1}{\sqrt{1+4x^2}}\left(\mathbf{E}_x - 2x\mathbf{E}_y\right), \\
\mathbf{e}_n &= \frac{-1}{\sqrt{1+4x^2}}\left(\mathbf{E}_y + 2x\mathbf{E}_x\right), \\
\mathbf{e}_b &= -\mathbf{E}_z.
\end{aligned}$$

The kinematics of the particle is given by the formulae in Section 3.4:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d^2s}{dt^2}\mathbf{e}_t + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{e}_n.$$

3.5.2 Forces

As shown in Figure 3.9, we next consider a free-body diagram of the particle. The forces acting on the particle are due to the gravitational force and the normal force $\mathbf{N} = N_n\mathbf{e}_n + N_b\mathbf{e}_b$:

$$\mathbf{F} = -mg\mathbf{E}_y + N_n\mathbf{e}_n + N_b\mathbf{e}_b.$$

Here, **N** is the force that the curve exerts on the particle.

3.5.3 Balance Law

Taking the components of $\mathbf{F} = m\mathbf{a}$ with respect to the Serret-Frenet triad, three scalar equations are obtained:

$$m\ddot{s} = \frac{2xmg}{\sqrt{1+4x^2}}, \qquad N_n = \frac{m\dot{s}^2}{\rho} - \frac{mg}{\sqrt{1+4x^2}}, \qquad N_b = 0.$$

3.5.4 Analysis

The last two of the above equations determine the normal force $\mathbf{N}$:

$$\mathbf{N} = \left(\frac{mv^2}{\rho} - \frac{mg}{\sqrt{1+4x^2}}\right)\mathbf{e}_n.$$

The first equation above determines the motion of the particle on the curve:

$$m\dot{v} = \frac{2xmg}{\sqrt{1+4x^2}}.$$

To proceed to describe this equation as a differential equation for $x(t)$, we note that

$$v = \frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \frac{dx}{dt}\sqrt{1+4x^2},$$
$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{4x}{\sqrt{1+4x^2}}\left(\frac{dx}{dt}\right)^2 + \sqrt{1+4x^2}\frac{d^2x}{dt^2}.$$

You should note that $v \neq dx/dt$. For the example at hand, we substitute these results into the differential equation for dv/dt. After some rearranging, we obtain the desired ordinary differential equation:

$$\frac{d^2x}{dt^2} = \frac{1}{1+4x^2}\left(2xg - 4x\left(\frac{dx}{dt}\right)^2\right).$$

Given the initial conditions $x(t_0)$ and $\dot{x}(t_0)$, the solution $x(t)$ of this equation can then be used to determine the motion $\mathbf{r}(t) = x(t)\mathbf{E}_x - x^2(t)\mathbf{E}_y$ of the particle and the force $\mathbf{N}(t)$ exerted by the curve on the particle.

The nonlinear ordinary differential equation governing $x(t)$ is formidable. Developing an analytical solution for $x(t)$ is well beyond the scope of an undergraduate engineering dynamics course. Instead, one is content with finding the speed v as a function of x. To this end, one uses the identity

$$a_t = \mathbf{a}\cdot\mathbf{e}_t = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{v}{\sqrt{1+4x^2}}\frac{dv}{dx}.$$

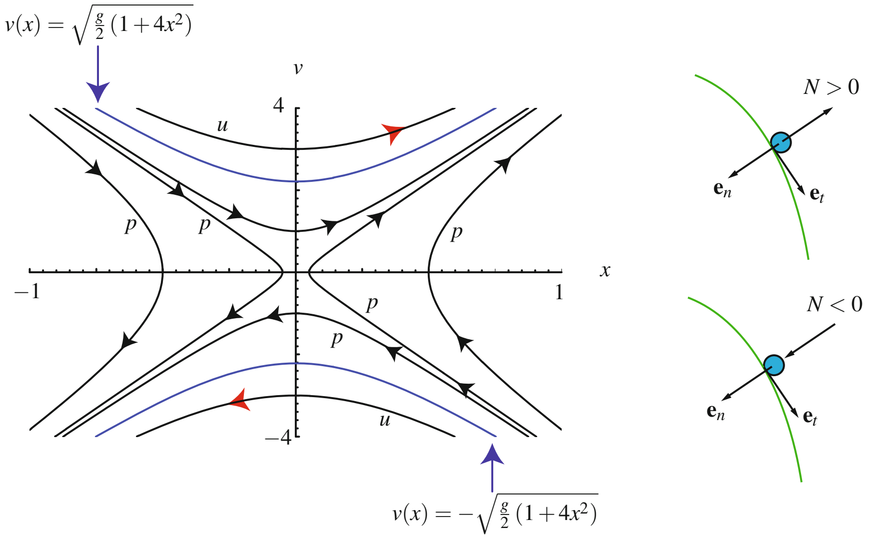

Fig. 3.10 *The speed* v *of a particle moving on the curve* $y = -x^2$ *shown in Figure 3.8. The arrows in this figure denote increasing time, and the curves formed by* $v(x) = \pm\sqrt{g\left(1+4x^2\right)/2}$ *divide regions where* $N > 0$ *and* $N < 0$*, where* $\mathbf{N} = -N\mathbf{e}_n$*. The six situations labeled* p *are physically possible because* $N > 0$*, whereas the two curves* $v(x)$ *labeled* u *are not.*

It follows that

$$v^2(x) = v^2(x_0) + 2\int_{x_0}^{x} a_t(u)\sqrt{1+4u^2}du.$$

For the example at hand,

$$a_t(x) = \frac{2xg}{\sqrt{1+4x^2}},$$

and hence,

$$v^2(x) = v^2(x_0) + \int_{x_0}^{x} 4ugdu = v^2(x_0) + 2g\left(x^2 - x_0^2\right).$$

Here, x_0 and $v(x_0)$ are given initial conditions. Because $y = -x^2$, you may have noticed that this is none other than a conservation of energy result (cf. the latter part of Section 5.7 of Chapter 5 where this problem is revisited).

We are now in a position to establish a criterion for the particle leaving the curve. One can use the previous equation and the expression for ρ to calculate the force $\mathbf{N}$ as a function of x:

$$\mathbf{N} = \left(\frac{mv^2}{\rho} - \frac{mg}{\sqrt{1+4x^2}}\right)\mathbf{e}_n = \left(\frac{2mv^2(x_0) - mg(1+4x_0^2)}{\left(\sqrt{1+4x^2}\right)^3}\right)\mathbf{e}_n.$$

Notice that if $v(x_0)$ is sufficiently large, then $\mathbf{N} \cdot \mathbf{e}_n \geq 0$ and the particle will not remain on the curve. Specifically, if

$$v^2(x_0) \geq \frac{g}{2}\left(1 + 4x_0^2\right),$$

then the particle immediately loses contact with the curve.

To illustrate the results for $v(x)$ and the loss of contact, we first use energy conservation, $v^2(x) = v^2(x_0) + 2g\left(x^2 - x_0^2\right)$, to show representative cases of $v(x)$ in Figure 3.10. We also show the curves where $v^2 = g\left(1 + 4x^2\right)/2$. You should notice from this figure that for certain motions of the particle, it moves up the curve $y = -x^2$ towards $x = 0$ but does not reach the summit. Instead, it stops, reverses its direction of motion, and starts moving down the curve. We leave it as an exercise to interpret the three other types of situations shown in Figure 3.10. You should note that for two of these situations, labeled u, the normal force needed to prevent the particle leaving the curve is a suction force and so the motions are not physically possible.

3.6 Summary

This chapter established the machinery needed to examine the dynamics of particles moving in a general manner in three-dimensional space. To this end, some results pertaining to curves in three-dimensional space were presented.

For a given space curve, the Serret-Frenet basis vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$ form a right-handed orthonormal basis at each point of the curve. The three vectors are defined by

$$\mathbf{e}_t = \frac{d\mathbf{r}}{ds}, \qquad \kappa\mathbf{e}_n = \frac{d^2\mathbf{r}}{ds^2}, \qquad \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n.$$

The rate of change of these vectors as the arc-length parameter s of the curve varies is given by the Serret-Frenet relations:

$$\frac{d\mathbf{e}_t}{ds} = \kappa\mathbf{e}_n, \qquad \frac{d\mathbf{e}_n}{ds} = -\kappa\mathbf{e}_t + \tau\mathbf{e}_b, \qquad \frac{d\mathbf{e}_b}{ds} = -\tau\mathbf{e}_n.$$

These relations and the Serret-Frenet basis vectors were illustrated for the case of a plane curve, a straight line, a circle, a space curve parametrized by x, and a circular helix. For three of these examples, it was convenient to describe the Serret-Frenet basis vectors, torsion τ, and curvature κ as functions of x rather than s. In a similar manner, the variable θ was used for the circle and circular helix.

To use these results in particle dynamics, the path of the particle is identified with a space curve. Then, we showed that

$$\begin{aligned}
\mathbf{v} &= \dot{s}\frac{d\mathbf{r}}{ds} = \dot{s}\mathbf{e}_t = v\mathbf{e}_t, \\
\mathbf{a} &= \dot{s}^2\frac{d^2\mathbf{r}}{ds^2} + \ddot{s}\frac{d\mathbf{r}}{ds} = \kappa\dot{s}^2\mathbf{e}_n + \ddot{s}\mathbf{e}_t = \kappa v^2\mathbf{e}_n + \dot{v}\mathbf{e}_t.
\end{aligned}$$

In these equations, $v = \dot{s}$. Using the Serret-Frenet basis vectors, the balance of linear momentum can be described by three equations:

$$F_t = \mathbf{F} \cdot \mathbf{e}_t = m\dot{v}, \qquad F_n = \mathbf{F} \cdot \mathbf{e}_n = m\kappa v^2, \qquad F_b = \mathbf{F} \cdot \mathbf{e}_b = 0.$$

These equations were illustrated using the example of a particle moving on a smooth fixed curve.

3.7 Exercises

The following short exercises are intended to assist you in reviewing the material in the present chapter.

3.1. For the space curve $\mathbf{r} = x\mathbf{E}_x + ax\mathbf{E}_y$, show that

$$\mathbf{e}_t = \frac{1}{\sqrt{1+a^2}} \left(\mathbf{E}_x + a\mathbf{E}_y \right), \qquad \kappa = 0.$$

In addition, show that $\mathbf{e}_n$ is any vector perpendicular to $\mathbf{e}_t$, for example,

$$\mathbf{e}_n = \frac{1}{\sqrt{1+a^2}} \left(-a\mathbf{E}_x + \mathbf{E}_y \right).$$

Why is the torsion τ of this curve 0?

3.2. Calculate the Serret-Frenet basis vectors for the space curve

$$\mathbf{r} = x\mathbf{E}_x + \frac{x^3}{3}\mathbf{E}_y.$$

It is convenient to describe these vectors as functions of x. In addition, show that the arc-length parameter s as a function of x is given by

$$s = s(x) = \int_{x_0}^{x} \sqrt{1+u^4}\,du + s(x_0).$$

3.3. Consider the space curve

$$\mathbf{r} = x\mathbf{E}_x + \frac{x^3}{3}\mathbf{E}_y + \frac{x^2}{2}\mathbf{E}_z.$$

Calculate the Serret-Frenet basis vectors for this curve. Again, it is convenient to describe these vectors as functions of x. In addition, show that the arc-length parameter s as a function of x is given by

$$s = s(x) = \int_{x_0}^{x} \sqrt{1+u^4+u^2}\,du + s(x_0).$$

3.4. Consider the plane circle $\mathbf{r} = 10\mathbf{e}_r$. Show that $\mathbf{e}_t = \mathbf{e}_\theta$, $\mathbf{e}_n = -\mathbf{e}_r$, $\kappa = 0.1$, $\rho = 10$, $\tau = 0$, and $\mathbf{e}_b = \mathbf{E}_z$.

3.5. Calculate the Serret-Frenet basis vectors for the circular helix $\mathbf{r} = 10\mathbf{e}_r + 10\theta\mathbf{E}_z$. In addition, show that $s(\theta) = 10\sqrt{2}(\theta - \theta_0) + s(\theta_0)$ and $\kappa = \tau = 1/20$.

3.6. Starting from $\mathbf{r} = \mathbf{r}(s(t))$, show that $\mathbf{v} = v\mathbf{e}_t$ and $\mathbf{a} = \dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n$.

3.7. A section of track of a roller coaster can be defined using the space curve $\mathbf{r} = x\mathbf{E}_x + f(x)\mathbf{E}_y$. The measurement system used to determine the speed of a trolley moving on this track measures $x(t)$ rather than $s(t)$. Consequently, in order to establish and verify the equations of motion of the trolley, it is desirable to know $\dot{x}$ and $\ddot{x}$ in terms of $\dot{s}$ and $\ddot{s}$. Starting from the definition $\dot{s} = \|\mathbf{v}\|$, show that

$$\dot{s} = \sqrt{1 + \left(\frac{df}{dx}\right)^2}\,\dot{x}.$$

Using this result and the chain rule, show that

$$\ddot{s} = \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2}}\left(\frac{df}{dx}\frac{d^2f}{dx^2}\dot{x}^2 + \left(1 + \left(\frac{df}{dx}\right)^2\right)\ddot{x}\right).$$

3.8. Using the results of Exercise 3.7 and the expressions for the Serret-Frenet basis vectors recorded in Section 3.3.1, write out the equations governing the motion of the trolley. You should assume that a gravitational force $-mg\mathbf{E}_y$ acts on the trolley while it is moving on the track, and model the trolley as a particle of mass m.

3.9. Consider a particle of mass m moving on a circular helix $\mathbf{r} = R\mathbf{e}_R + \alpha R\theta\mathbf{E}_z$. A gravitational force $-mg\mathbf{E}_z$ acts on the particle. Show that

$$s(\theta) = R\sqrt{1+\alpha^2}(\theta - \theta_0) + s(\theta_0), \qquad \ddot{s} = R\sqrt{1+\alpha^2}\,\ddot{\theta}.$$

Using $\mathbf{F} = m\mathbf{a}$ and the results of Section 3.4, show that

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{e}_t &= -\frac{mg\alpha}{\sqrt{1+\alpha^2}} = mR\sqrt{1+\alpha^2}\,\ddot{\theta},\\
\mathbf{F}\cdot\mathbf{e}_n &= N_n = mR\dot{\theta}^2,\\
\mathbf{F}\cdot\mathbf{e}_b &= N_b - \frac{mg}{\sqrt{1+\alpha^2}}.
\end{aligned}$$

Here, $N_n\mathbf{e}_n + N_b\mathbf{e}_b$ is the normal force exerted by the curve on the particle.

3.10. Show that the polar coordinate θ of the particle discussed in Exercise 3.9 is

$$\theta(t) = -\frac{g\alpha}{2R(1+\alpha^2)}(t-t_0)^2 + \dot{\theta}_0(t-t_0) + \theta_0.$$

For various initial conditions θ_0 and $\dot{\theta}_0$, discuss the motion of the particle on the helix. You should notice the similarities between the results of this exercise and the projectile problem discussed in Section 1.6 of Chapter 1.

3.11. The equation of a clothoid is given by

$$x(u) = a\sqrt{\pi}\int_0^{u/\sqrt{\pi}} \cos\left(\frac{\pi z^2}{2}\right) dz,$$
$$y(u) = a\sqrt{\pi}\int_0^{u/\sqrt{\pi}} \sin\left(\frac{\pi z^2}{2}\right) dz,$$

where $a > 0$ is a constant (see Figure 3.3). Show that the arc-length parameter s of the clothoid is governed by the equation

$$\frac{ds}{du} = a.$$

Verify that the curvature is given by the simple expression $\kappa = u/a$, and compute the Serret-Frenet triad for the clothoid.

Chapter 4
Friction Forces and Spring Forces

TOPICS

Two types of forces are discussed in this chapter: friction forces and spring forces. We start with the former and consider a simple classical experiment. Based on this experiment, general (coordinate-free) expressions for friction forces are obtained. The chapter closes with the corresponding developments for a spring force.

The reason for including a separate chapter on these two types of forces is that I have found that they present the most difficulty to students when they are formulating problems. In particular, many students have the impression that if they do not correctly guess the direction of spring and friction forces in their free-body diagram, then they will obtain the wrong answer. The coordinate-free formulation of these forces in this chapter bypasses this issue.

4.1 An Experiment on Friction

Most theories of friction forces arise from studies by the French scientist Charles Augustin Coulomb.[1] Here, we review a simple experiment that is easily replicated (at least qualitatively) using a blackboard eraser, some weights, and a table.

As shown in Figure 4.1, consider a block of mass m that is initially at rest on a horizontal plane. A force $P\mathbf{E}_x$ acts on the block.

Upon increasing P, several observations can be made:

(a) For small values of P, the block remains at rest.
(b) Beyond a critical value $P = P^*$, the block starts to move.
(c) Once in motion, a constant value of $P = P^{**}$ is required to move the block at a constant speed.
(d) Both P^* and P^{**} are proportional to the magnitude of the normal force $\mathbf{N}$.

[1] The most-cited reference to his work is his prize-winning paper [18], which was published in 1785. Accounts of Coulomb's work on friction are contained in Dugas [23] and Heyman [35].

O.M. O'Reilly, *Engineering Dynamics: A Primer*, DOI 10.1007/978-1-4419-6360-4_4,

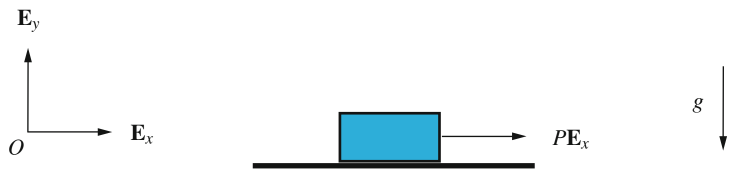

Fig. 4.1 *A block on a rough horizontal plane.*

Let's now analyze this experiment. We model the block as a particle of mass m, and set

$$\mathbf{r} = x\mathbf{E}_x + y_0\mathbf{E}_y + z_0\mathbf{E}_z,$$

where y_0 and z_0 are constants. We leave it as an exercise to derive expressions for $\mathbf{v}$ and $\mathbf{a}$.

The free-body diagram of the particle is shown in Figure 4.2. In this figure, $\mathbf{F}_f = F_{f_x}\mathbf{E}_x + F_{f_z}\mathbf{E}_z$ is the friction force exerted by the surface on the particle and $\mathbf{N} = N\mathbf{E}_y$ is the normal (or reaction) force exerted by the surface on the particle.

From $\mathbf{F} = m\mathbf{a}$, we obtain three equations:

$$\begin{aligned} F_{f_x} + P &= m\ddot{x}, \\ N - mg &= 0, \\ F_{f_z} &= 0. \end{aligned}$$

For the static case, $\ddot{x} = 0$, and from $\mathbf{F} = m\mathbf{a}$ we find that $\mathbf{F}_f = -P\mathbf{E}_x$. As noted previously, as the magnitude of $\mathbf{P}$ is increased beyond a critical value P^*, the block starts to move. The critical value of P^* is proportional to the magnitude of the normal force $\mathbf{N}$. We denote this constant of proportionality by the coefficient of static friction μ_s. Hence, $\left|F_{f_x}\right| \leq \mu_s |N| = \mu_s mg$ and $P^* = \mu_s mg$. In summary, for the static case, $\mathbf{F}_f = -P\mathbf{E}_x$ until $P = \mu_s mg$. As P increases beyond this value, then the block moves and the friction force is no longer equal to $-P\mathbf{E}_x$.

When the block starts moving, then the friction force opposes the motion. As a result, its direction is opposite to $\mathbf{v}$. Furthermore, as noted above, its magnitude is proportional to the magnitude of $\mathbf{N}$. The constant of proportionality is denoted by μ_d, the coefficient of dynamic friction. Hence, assuming that the block moves to the

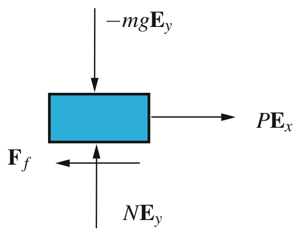

Fig. 4.2 *Free-body diagram of the block.*

right, $\mathbf{F}_f = -\mu_d|N|\mathbf{E}_x = -\mu_d mg\mathbf{E}_x$. If the block moves at constant speed, then we find from $\mathbf{F} = m\mathbf{a}$ that $P^{**} = \mu_d mg$.

The coefficients of static friction and dynamic friction depend on the nature of the surfaces of the horizontal surface and the block. They must be determined experimentally.

4.2 Static and Dynamic Coulomb Friction Forces

Among other assumptions, the previous developments assumed that the horizontal surface was flat and stationary. For many problems, our developments are insufficient and need to be generalized in several directions:

(a) Cases where the surface on which the particle lies is curved.
(b) Cases where the particle is moving on a space curve.
(c) Cases where the particle moves on a space curve or surface which is in motion.

It is to these cases that we now turn. The theory that we present here is not universally applicable, although it is used extensively in engineering.[2] For instance, in theories proposed to explain the squealing induced by brake pads pressing on rotors in automobiles, the Coulomb friction theory is used extensively [41]. On the other hand, this friction theory is generally not used in modeling the contact forces between tires and the surface of the road.

To proceed, the position vector and absolute velocity vector of the particle are denoted by $\mathbf{r}$ and $\mathbf{v}$, respectively. If the particle is in motion on a space curve, then the velocity vector of the point of the space curve that is in contact with the particle is denoted by $\mathbf{v}_c$. Similarly, if the particle is in motion on a surface, then the velocity vector of the point of the surface that is in contact with the particle is denoted by $\mathbf{v}_s$.

4.2.1 A Particle on a Surface

Here, we assume that the particle is moving on a surface. Referring to Figure 4.3, at the point P of the surface that is in contact with the particle, we assume that there is a well-defined unit normal vector $\mathbf{n}$. At this point of contact one also has a pair of unit tangent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. We choose these tangent vectors such that $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}\}$ is a right-handed basis for Euclidean three-space.

It is convenient to consider some examples. First, if the particle is moving on a horizontal plane, then $\mathbf{n} = \mathbf{E}_z$, $\mathbf{t}_1 = \mathbf{E}_x$, and $\mathbf{t}_2 = \mathbf{E}_y$. Another example, which we

[2] For further information on the limitations of the Coulomb friction theory, we refer the reader to Rabinowicz [62] and Ruina [67].

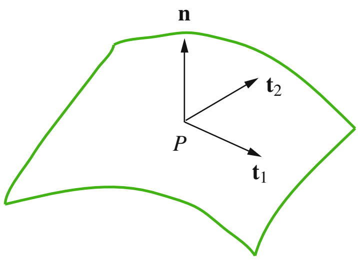

Fig. 4.3 *The pair of tangent vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *and normal vector* $\mathbf{n}$ *at a point* P *of a surface.*

examine later in Section 4.5, is a particle on the inner surface of a cone $z = r\tan(\alpha)$. Here, α is the angle of inclination of the conical surface. In this case, we have

$$\mathbf{n} = \cos(\alpha)\mathbf{E}_z - \sin(\alpha)\mathbf{e}_r, \quad \mathbf{t}_1 = \cos(\alpha)\mathbf{e}_r + \sin(\alpha)\mathbf{E}_z, \quad \mathbf{t}_2 = \mathbf{e}_\theta.$$

Notice how these vectors have been normalized so as to have unit magnitude.

Recall that $\mathbf{v}_s$ denotes the absolute velocity vector of the point of contact P. Then, the velocity vector of the particle relative to P is

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_s = v_1\mathbf{t}_1 + v_2\mathbf{t}_2.$$

If $\mathbf{v}_{\text{rel}} = \mathbf{0}$, then the particle is said to be stationary relative to the surface. Specifically, we have

static friction when $\mathbf{v}_{\text{rel}} = \mathbf{0}$ and dynamic friction when $\mathbf{v}_{\text{rel}} \neq \mathbf{0}$.

The force exerted by the surface on the particle is composed of two parts: the normal (or reaction) force $\mathbf{N}$ and the friction force $\mathbf{F}_f$. For both the static and dynamic cases,

$$\mathbf{N} = N\mathbf{n}.$$

Furthermore, N is indeterminate. It can be found only from $\mathbf{F} = m\mathbf{a}$. The static friction force is

$$\mathbf{F}_f = F_{f_1}\mathbf{t}_1 + F_{f_2}\mathbf{t}_2,$$

where F_{f_1} and F_{f_2} are also indeterminate. The amount of static friction available is limited by the coefficient of static friction:

$$\left\|\mathbf{F}_f\right\| \leq \mu_s \left\|\mathbf{N}\right\|.$$

If this criterion fails, then the particle will move relative to the surface. The friction force in this case is dynamic:

$$\mathbf{F}_f = -\mu_d \left\|\mathbf{N}\right\| \frac{\mathbf{v}_{\text{rel}}}{\left\|\mathbf{v}_{\text{rel}}\right\|}.$$

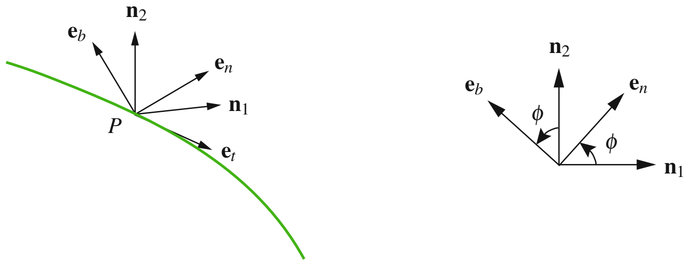

Fig. 4.4 *The tangent vector* $\mathbf{e}_t$ *and two normal vectors* $\mathbf{n}_1$ *and* $\mathbf{n}_2$ *at a point P of a curve. The inset figure shows the pair of normal vectors lying in the plane formed by* $\mathbf{e}_n$ *and* $\mathbf{e}_b$.

You should notice that this force opposes the motion of the particle relative to the surface.

4.2.2 A Particle on a Space Curve

Here, we assume that the particle is moving on a curve. At the point P of the curve that is in contact with the particle, we assume that there is a well-defined unit tangent vector $\mathbf{e}_t$ (see Figure 4.4). At this point of contact, one also has two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. We choose these vectors such that $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{e}_t\}$ is a right-handed basis for Euclidean three-space. The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ lie in the plane spanned by $\mathbf{e}_n$ and $\mathbf{e}_b$. For instance, if the particle is moving on a horizontal line, then we can choose $\mathbf{e}_t = \mathbf{E}_x$, $\mathbf{n}_1 = \mathbf{E}_y$, and $\mathbf{n}_2 = \mathbf{E}_z$.

Recall that $\mathbf{v}_c$ denotes the absolute velocity vector of the point of contact P. Then, the velocity vector of the particle relative to P is

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_c = v_t \mathbf{e}_t .$$

If $\mathbf{v}_{\text{rel}} = \mathbf{0}$, then the particle is said to be stationary relative to the curve. Specifically, we again have that

$$\text{static friction when } \mathbf{v}_{\text{rel}} = \mathbf{0} \text{ and dynamic friction when } \mathbf{v}_{\text{rel}} \neq \mathbf{0}.$$

The force exerted by the curve on the particle is composed of two parts: the normal (or reaction) force $\mathbf{N}$ and the friction force $\mathbf{F}_f$. For both the static and dynamic cases,

$$\mathbf{N} = N_1 \mathbf{n}_1 + N_2 \mathbf{n}_2 .$$

Furthermore, N_1 and N_2 are indeterminate. They can be found only from $\mathbf{F} = m\mathbf{a}$. The static friction force is

$$\mathbf{F}_f = F_f \mathbf{e}_t ,$$

where F_f is also indeterminate. The amount of static friction available is limited by the coefficient of static friction:

$$\left\|\mathbf{F}_f\right\| \leq \mu_s \left\|\mathbf{N}\right\|.$$

Specifically,

$$\left|F_f\right| \leq \mu_s \sqrt{N_1^2 + N_2^2}.$$

If this criterion fails, then the particle will move relative to the curve. The friction force in this case is dynamic:

$$\begin{aligned}\mathbf{F}_f &= -\mu_d \left\|\mathbf{N}\right\| \frac{\mathbf{v}_{\text{rel}}}{\left\|\mathbf{v}_{\text{rel}}\right\|} \\ &= -\mu_d \sqrt{N_1^2 + N_2^2}\,\frac{v_t}{\left|v_t\right|}\mathbf{e}_t.\end{aligned}$$

You should notice that this force opposes the motion of the particle relative to the curve.

4.2.3 Additional Comments

In the static friction case, $\mathbf{v}_{\text{rel}} = \mathbf{0}$ and the particle is stuck on the surface or curve. For these instances, both the friction force $\mathbf{F}_f$ and the normal force $\mathbf{N}$ are unknown. However, because the motion of the surface/curve is known, the motion of the particle is also known and $\mathbf{a}$ can be determined. Hence, $\mathbf{F} = m\mathbf{a}$ provides three equations to determine these forces.

An error many students make is setting the static friction force F_{f_x}, say, equal to $\mu_s N$ or, worse, $\mu_s mg$. Setting the static friction force equal to one of its maximum values is generally not valid. The easiest example to explain this error is shown in Figure 4.5. In this figure, two examples of the friction force for a block that is initially placed at rest on a plane are shown. If the plane were horizontal, then it would be easy to show that $\mathbf{F}_f = \mathbf{0}$. As the angle of inclination ϕ of the plane is increased, the magnitude of the friction force increases. Eventually, there is insufficient static friction to hold the block at rest and it starts moving. Once in motion, the friction force changes to dynamic Coulomb friction and the block accelerates down the incline.

The careful reader will have noted that we are using $\mathbf{e}_t$ to denote the tangent vector to a moving curve. However, our previous developments in Chapter 3 were limited to a fixed curve. As mentioned there, they can be extended to a moving curve. Specifically, let $\mathbf{p}$ denote the position vector to any point on a moving space curve. This position vector depends on the arc-length parameter and time:

$$\mathbf{p} = \tilde{\mathbf{p}}(s,t).$$

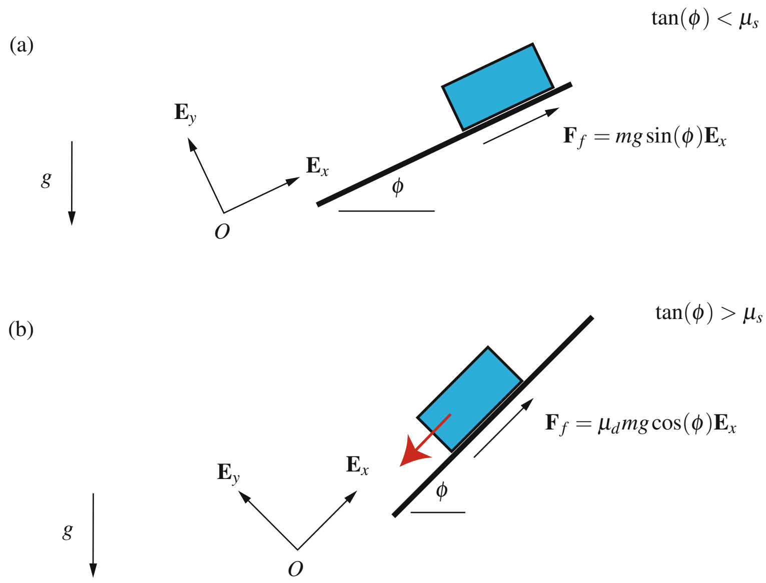

Fig. 4.5 *A block on a rough plane at various angles of inclination: (a)* $\tan(\phi) < \mu_s$ *and the block remains at rest and (b)* $\tan(\phi) > \mu_s$ *and there is insufficient static friction to prevent motion of the block.*

One now has derivatives featuring in the definitions of the Serret-Frenet triad are partial derivatives with respect to s:

$$\mathbf{e}_t = \tilde{\mathbf{e}}_t(s,t) = \frac{\partial \mathbf{p}}{\partial s}, \qquad \kappa \mathbf{e}_n = \frac{\partial \mathbf{e}_t}{\partial s}, \qquad \mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n.$$

In essence one is *freezing* time and evaluating the Serret-Frenet triad for the *frozen* curve. Another point of interest is that the velocity $\mathbf{v}_c$ discussed earlier is equal to $\partial \mathbf{p}/\partial t$. It is interesting to note that these developments are used in theories of rods (see, e.g., Antman [1] or Love [46]).

4.3 A Particle on a Rough Moving Plane

To illustrate the previous developments, we consider the problem of a particle moving on a rough horizontal plane as shown in Figure 4.6. Every point on this plane is assumed to be moving with a velocity $\mathbf{v}_s = v_s \mathbf{E}_z$. A vertical gravitational force also acts on the particle.

For this problem,

$$\mathbf{t}_1 = \mathbf{E}_x, \qquad \mathbf{t}_2 = \mathbf{E}_y, \qquad \mathbf{n} = \mathbf{E}_z.$$

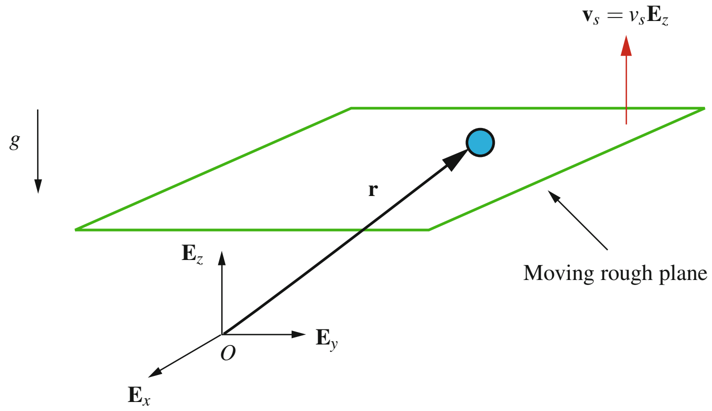

Fig. 4.6 *A particle moving on a horizontal plane.*

Furthermore, the velocity vector of the particle is

$$\mathbf{v} = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y + v_s\mathbf{E}_z.$$

The relative velocity vector of the particle is

$$\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_s = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y.$$

We leave it as an exercise to draw the free-body diagrams of the particle for the static and dynamic cases.

Turning to the results for the static case, we have

$$\mathbf{F} = N\mathbf{E}_z + F_{f_x}\mathbf{E}_x + F_{f_y}\mathbf{E}_y - mg\mathbf{E}_z = m\mathbf{a} = m\dot{v}_s\mathbf{E}_z.$$

It follows from these equations that

$$\mathbf{F}_f = \mathbf{0}, \qquad \mathbf{N} = m(g + \dot{v}_s)\mathbf{E}_z.$$

As expected, the friction force is zero in this case.

Alternatively for the dynamic case we have

$$\mathbf{F} = N\mathbf{E}_z - \mu_d\,|N|\,\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|} - mg\mathbf{E}_z = m\mathbf{a}.$$

From these three scalar equations, we find that the normal force is

$$\mathbf{N} = m(g + \dot{v}_s)\mathbf{E}_z.$$

In addition, two ordinary differential equations of the motion of the particle relative to the surface are obtained:

$$m\ddot{x} = -\mu_d \left| m(g + \dot{v}_s) \right| \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \qquad m\ddot{y} = -\mu_d \left| m(g + \dot{v}_s) \right| \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.$$

As expected, these differential equations are valid only when $\|\mathbf{v}_{\mathrm{rel}}\|$ is nonzero.

4.4 Hooke's Law and Linear Springs

The classical result on linear springs is due to a contemporary of Isaac Newton, Robert Hooke (1635–1703), who announced his result in the form of the anagram *ceiiinosssttuu* in 1660. He later published his result in his work *Lectures de Potentia restitutiva, or, Of spring explaining the power of springing bodies* in 1678:

Ut tensio sic vis

which tranlates to "the power of any spring is in the same proportion with the tension thereof."[3] His result is often known as Hooke's law.

As with Coulomb's work on friction, Hooke's law is based on experimental evidence. It is not valid in all situations. For instance, it implies that it is possible to extend a spring as much as desired without the spring breaking, which is patently not true. However, for many applications, where the change in the spring's length is small, it is a valid and extremely useful observation.

Referring to Figure 4.7, we wish to develop a general result for the force exerted by a (massless) spring on a particle when the motion of the particle changes the length of the spring. One end of the spring is attached to a point A which has a position vector $\mathbf{r}_A$. Its other end is attached to a mass particle whose position vector is $\mathbf{r}$. We confine our attention to a linear spring and denote its stiffness by K. That is, we assume Hooke's law is valid.

The force generated by the spring is assumed to be linearly proportional to its extension/compression. Here, the change in length of the spring is

$$\|\mathbf{r} - \mathbf{r}_A\| - L,$$

where L is the unstretched length of the spring. If this number is positive, then the spring is extended. The magnitude of the spring force $\mathbf{F}_s$ is

$$\|\mathbf{F}_s\| = \left| K\left(\|\mathbf{r} - \mathbf{r}_A\| - L\right) \right|.$$

This is a statement of Hooke's law.

[3] From the *Historical Introduction* to Love [46]. In modern terminology, *power* is force and *tension* is extension.

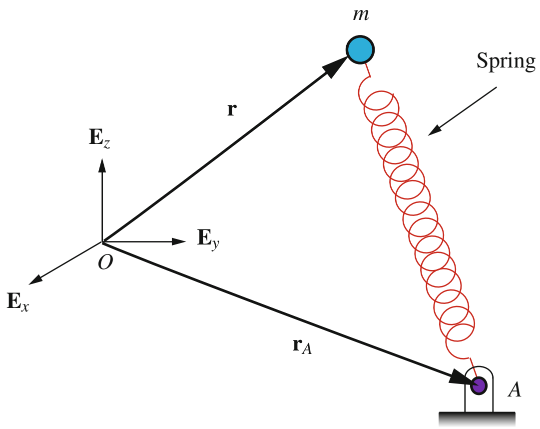

Fig. 4.7 *A spring attached to a mass particle.*

It remains to determine the direction of $\mathbf{F}_s$. First, suppose the spring is extended. Then the force $\mathbf{F}_s$ will attempt to pull the particle towards A. In other words, its direction is

$$-\frac{\mathbf{r}-\mathbf{r}_A}{\|\mathbf{r}-\mathbf{r}_A\|}.$$

Combining this observation with the result on the magnitude of the spring force, and noting that the extension is positive, we arrive at the expression

$$\mathbf{F}_s = -K\left(\|\mathbf{r}-\mathbf{r}_A\| - L\right)\frac{\mathbf{r}-\mathbf{r}_A}{\|\mathbf{r}-\mathbf{r}_A\|}.$$

On the other hand, if the spring is compressed, then this force will attempt to push the particle away from A. As a result its direction will be

$$\frac{\mathbf{r}-\mathbf{r}_A}{\|\mathbf{r}-\mathbf{r}_A\|}.$$

Because the change in length of the spring force is negative, we find that the magnitude of the spring force in this case is

$$\|\mathbf{F}_s\| = -K\left(\|\mathbf{r}-\mathbf{r}_A\| - L\right).$$

Consequently, the spring force when the spring is compressed is

$$\mathbf{F}_s = -K\left(\|\mathbf{r}-\mathbf{r}_A\| - L\right)\frac{\mathbf{r}-\mathbf{r}_A}{\|\mathbf{r}-\mathbf{r}_A\|}.$$

It should be clear that the final expressions for $\mathbf{F}_s$ we have obtained for the extended and compressed springs are identical. In summary, for a spring of stiffness

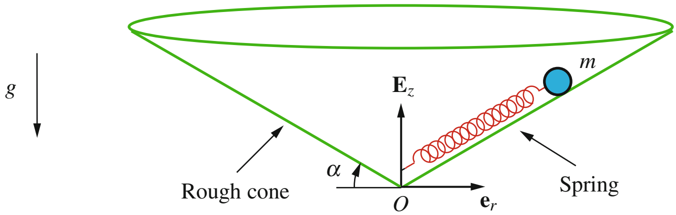

Fig. 4.8 *A particle moving on a rough spinning cone.*

K and unstretched length L, the force exerted by the spring on the particle is

$$\mathbf{F}_s = -K\left(\|\mathbf{r}-\mathbf{r}_A\| - L\right)\frac{\mathbf{r}-\mathbf{r}_A}{\|\mathbf{r}-\mathbf{r}_A\|}.$$

If, for a specific problem, one can choose the point A to be the origin, then the expression for $\mathbf{F}_s$ simplifies considerably:

$$\mathbf{F}_s = -K\left(\|\mathbf{r}\| - L\right)\frac{\mathbf{r}}{\|\mathbf{r}\|}.$$

4.5 A Particle on a Rough Spinning Cone

We now consider the dynamics of a particle on a rough circular cone $z = r\tan(\alpha)$. The cone is rotating about its axis of symmetry with an angular speed $\Omega = \Omega(t)$. As shown in Figure 4.8, the particle is attached to the apex of the cone by a spring of stiffness K and unstretched length L.

In what follows, we seek to determine the differential equations governing the motion of the particle and, in the event that the particle is not moving, the force exerted by the surface on the particle. The problem presented here is formidable; it has spring forces, friction, and a nontrivial surface. However, various special cases of this system arise in many problems in mechanics. For instance, by setting $\alpha = 0^\circ$ or 90°, one has the problem of a particle on a cylinder or horizontal plane, respectively. Representative trajectories of the particle for the case when the cone is smooth are shown in Figure 4.9.

4.5.1 *Kinematics*

We choose our origin to coincide with the fixed apex of the cone. Neglecting the thickness of the spring, this point also coincides with the point of attachment of the spring to the apex of the cone. It is convenient to use a cylindrical polar coordinate

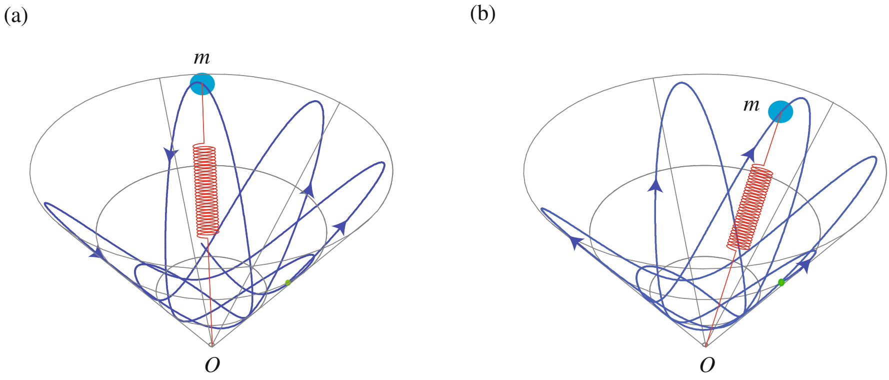

Fig. 4.9 *Examples of the trajectories of a particle moving on a smooth cone. The particle is attached to the apex O of the cone by a linear spring and a vertical gravitational force also acts on the particle. For (a)* $\dot{\theta}(0) > 0$ *and for (b)* $\dot{\theta}(0) < 0$.

system to describe the kinematics of this problem:

$$\mathbf{r} = r\mathbf{e}_r + r\tan(\alpha)\mathbf{E}_z.$$

Differentiating this position vector, we find that

$$\mathbf{v} = \dot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + r\dot{\theta}\mathbf{e}_\theta,$$
$$\mathbf{a} = \ddot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r.$$

We also note that $\|\mathbf{r}\| = r/\cos(\alpha)$.

The position vector of the point of contact P of the particle with the cone is the same as that for the particle. It follows that, at P, the normal and tangential vectors are

$$\mathbf{n} = \cos(\alpha)\mathbf{E}_z - \sin(\alpha)\mathbf{e}_r, \qquad \mathbf{t}_1 = \cos(\alpha)\mathbf{e}_r + \sin(\alpha)\mathbf{E}_z, \qquad \mathbf{t}_2 = \mathbf{e}_\theta.$$

Turning to the velocity vector of the point P, it is easily seen that this vector is

$$\mathbf{v}_s = r\Omega\mathbf{e}_\theta.$$

Consequently,

$$\mathbf{v}_{\text{rel}} = \dot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + r(\dot{\theta} - \Omega)\mathbf{e}_\theta.$$

You should notice that, in general, $\dot{\theta} \neq \Omega$. However, if $\mathbf{v}_{\text{rel}}$ is zero, then $\mathbf{a}$ simplifies considerably.

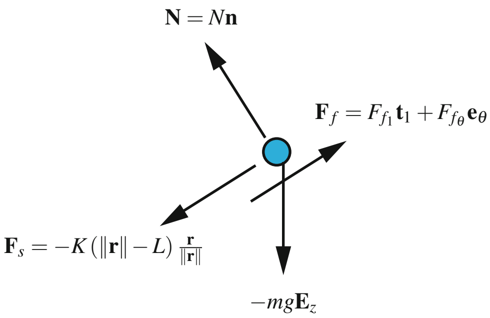

Fig. 4.10 *Free-body diagram of the particle.*

4.5.2 Forces

The free-body diagram for this problem is shown in Figure 4.10. Because we have chosen the origin to be the point of attachment of the spring, the spring force has the representation

$$\mathbf{F}_s = -K(\|\mathbf{r}\| - L)\frac{\mathbf{r}}{\|\mathbf{r}\|} = -K(r\sec(\alpha) - L)\mathbf{t}_1.$$

Notice that we are taking the general expression for the spring force that was established in Section 4.4 and substituting for $\mathbf{r}$. Alternatively, one can derive the second expression for the spring force above from scratch. We follow the (far easier) procedure of substituting directly into the general expression throughout the remainder of this book. Furthermore, the normal and friction forces acting on the particle are

$$\mathbf{N} = N\mathbf{n}, \qquad \mathbf{F}_f = F_{f_1}\mathbf{t}_1 + F_{f_\theta}\mathbf{e}_\theta.$$

The resultant force acting on the particle is

$$\mathbf{F} = N\mathbf{n} + \left(F_{f_1} - K(r\sec(\alpha) - L)\right)\mathbf{t}_1 + F_{f_\theta}\mathbf{e}_\theta - mg\mathbf{E}_z.$$

4.5.3 Balance Law

It is convenient to take the $\mathbf{t}_1$, $\mathbf{e}_\theta$, and $\mathbf{n}$ components of $\mathbf{F} = m\mathbf{a}$:

$$\begin{aligned}
F_{f_1} - K(r\sec(\alpha) - L) - mg\sin(\alpha) &= m\left(\ddot{r}\sec(\alpha) - r\dot{\theta}^2\cos(\alpha)\right), \\
F_{f_\theta} &= mr\ddot{\theta} + 2m\dot{r}\dot{\theta}, \\
N - mg\cos(\alpha) &= mr\dot{\theta}^2\sin(\alpha).
\end{aligned}$$

We have omitted several algebraic details that were used to arrive at the final form of these equations.

4.5.4 Analysis

From the three equations above, we find that the normal force **N** is

$$\mathbf{N} = mg\cos(\alpha)\mathbf{n} + mr\dot{\theta}^2\sin(\alpha)\mathbf{n}.$$

If the particle is not moving relative to the surface, then $\dot{\theta} = \Omega$ and $r = r_0$. As a result, the expression for **N** becomes

$$\mathbf{N} = \left(mg\cos(\alpha) + mr_0\Omega^2\sin(\alpha)\right)\mathbf{n}.$$

In both cases, this force always points in the direction of **n**, so the particle does not lose contact with the surface.

Turning to the static friction case, we find from the remaining two equations that the friction force is

$$\mathbf{F}_f = \left(K\left(r_0\sec(\alpha) - L\right) + mg\sin(\alpha) - mr_0\Omega^2\cos(\alpha)\right)\mathbf{t}_1 + mr_0\dot{\Omega}\mathbf{e}_\theta,$$

where $\mathbf{r}_0 = r_0\mathbf{e}_r + r_0\tan(\alpha)\mathbf{E}_z$. You should notice that if Ω is constant, then the static friction force has no component in the $\mathbf{e}_\theta$ direction. The static friction force is limited by the static friction criterion:

$$\left\|\mathbf{F}_f\right\| \leq \mu_s\|\mathbf{N}\| = \mu_s\left|mg\cos(\alpha) + mr_0\Omega^2\sin(\alpha)\right|.$$

When this criterion fails, then the particle begins to move relative to the surface.

When the particle is in motion relative to the surface, the friction force is

$$\begin{aligned}\mathbf{F}_f &= -\mu_d\|\mathbf{N}\|\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|}\\ &= -\mu_d|mg\cos(\alpha) + mr\dot{\theta}^2\sin(\alpha)|\frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|},\end{aligned}$$

where

$$\mathbf{v}_{\text{rel}} = \dot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + r(\dot{\theta} - \Omega)\mathbf{e}_\theta.$$

Substituting for the components of the dynamic friction force in the two equations obtained from **F** = m**a**,

$$\begin{aligned}F_{f_1} - mg\sin(\alpha) - K\left(r\sec(\alpha) - L\right) &= m\left(\ddot{r}\sec(\alpha) - r\dot{\theta}^2\cos(\alpha)\right),\\ F_{f_\theta} &= mr\ddot{\theta} + 2m\dot{r}\dot{\theta},\end{aligned}$$

one arrives at a pair of ordinary differential equations for r and θ. The solution of these equations determines the motion of the particle relative to the surface. It is these equations that were integrated numerically with $F_{f_1} = F_{f_\theta} = 0$ to produce the results shown in Figure 4.9.

4.6 Summary

Two types of forces were discussed in this chapter: friction forces and spring forces. For the friction forces, two classes needed to be considered, namely, static friction forces and dynamic friction forces. In addition, it was necessary to consider the two cases of a particle moving relative to a curve and a particle moving relative to a surface.

For a particle moving relative to a curve, a triad $\{\mathbf{e}_t, \mathbf{n}_1, \mathbf{n}_2\}$ associated with each point of the curve was defined. It should be noted that the vectors $\mathbf{e}_n$ and $\mathbf{e}_b$ will lie in the plane spanned by $\mathbf{n}_1$ and $\mathbf{n}_2$. The velocity vector of the point of the curve that is in contact with the particle was denoted $\mathbf{v}_c$. Hence, the velocity vector of the particle relative to its point of contact with the curve was $\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_c = v_t \mathbf{e}_t$. The friction force in this case had one component, $\mathbf{F}_f = F_f \mathbf{e}_t$, whereas the normal force had two components: $\mathbf{N} = N_1 \mathbf{n}_1 + N_2 \mathbf{n}_2$.

For a particle moving relative to a surface, a triad $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$ associated with each point of the surface was defined. The velocity vector of the point of the surface that is in contact with the particle was denoted $\mathbf{v}_s$. Hence, the velocity vector of the particle relative to its point of contact with the surface was $\mathbf{v}_{\text{rel}} = \mathbf{v} - \mathbf{v}_s = v_1 \mathbf{t}_1 + v_2 \mathbf{t}_2$. The friction force in this case had two components, $\mathbf{F}_f = F_{f_1} \mathbf{t}_1 + F_{f_2} \mathbf{t}_2$, whereas the normal force had one component: $\mathbf{N} = N\mathbf{n}$.

For both cases, when $\mathbf{v}_{\text{rel}}$ is nonzero, the friction force is of the dynamic Coulomb friction type:

$$\mathbf{F}_f = -\mu_d \|\mathbf{N}\| \frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|}.$$

However, if there is no relative motion, then $\mathbf{v}_{\text{rel}} = \mathbf{0}$ and $\mathbf{F}_f$ is indeterminate. In other words, it must be calculated from $\mathbf{F} = m\mathbf{a}$. However, the magnitude of the friction force is limited by the static friction criterion $\left\|\mathbf{F}_f\right\| \leq \mu_s \|\mathbf{N}\|$.

The second force we examined was the spring force. We considered a linear spring of stiffness K and unstretched length L. One end of the spring was attached to a fixed point A and the other end was attached to a particle. Using Hooke's law, the spring force was shown to be

$$\mathbf{F}_s = -K\left(\|\mathbf{r} - \mathbf{r}_A\| - L\right) \frac{\mathbf{r} - \mathbf{r}_A}{\|\mathbf{r} - \mathbf{r}_A\|}.$$

Finally, several examples were presented to illustrate the use of the aforementioned expressions for the spring and friction forces.

4.7 Exercises

The following short exercises are intended to assist you in reviewing Chapter 4.

4.1. Consider a particle moving on a circle of radius R. The position vector of the particle is $\mathbf{r} = R\mathbf{e}_r$. Show that the dynamic friction force and normal force have the representations

$$\mathbf{F}_f = -\mu_d \|\mathbf{N}\| \frac{\dot{\theta}}{|\dot{\theta}|}\mathbf{e}_\theta, \qquad \mathbf{N} = N_r\mathbf{e}_r + N_z\mathbf{E}_z.$$

4.2. Consider a particle moving on a cylinder of radius R. The position vector of the particle is $\mathbf{r} = R\mathbf{e}_r + z\mathbf{E}_z$. Show that the dynamic friction force and normal force have the representations

$$\mathbf{F}_f = -\mu_d \|\mathbf{N}\| \frac{R\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{E}_z}{\sqrt{R^2\dot{\theta}^2 + \dot{z}^2}}, \qquad \mathbf{N} = N\mathbf{e}_r.$$

4.3. Consider a particle that is stationary on a rough circle of radius R. The position vector of the particle is $\mathbf{r} = R\mathbf{e}_r$. Show that the static friction force and normal force have the representations

$$\mathbf{F}_f = F_f\mathbf{e}_\theta, \qquad \mathbf{N} = N_r\mathbf{e}_r + N_z\mathbf{E}_z.$$

In addition, show that the static friction criterion for this problem is

$$|F_f| \leq \mu_s\sqrt{N_r^2 + N_z^2}.$$

Show that this inequality is equivalent to

$$-\mu_s\sqrt{N_r^2 + N_z^2} \leq F_f \leq \mu_s\sqrt{N_r^2 + N_z^2}.$$

4.4. Consider a particle that is stationary on a rough cylinder of radius R. The position vector of the particle is $\mathbf{r} = R\mathbf{e}_r + z\mathbf{E}_z$. Show that the static friction force and normal force have the representations

$$\mathbf{F}_f = F_{f_\theta}\mathbf{e}_\theta + F_{f_z}\mathbf{E}_z, \qquad \mathbf{N} = N\mathbf{e}_r.$$

In addition, show that the static friction criterion for this problem is

$$\sqrt{F_{f_\theta}^2 + F_{f_z}^2} \leq \mu_s\,|N|.$$

4.5. A particle of mass m is connected to a fixed point O by a spring of stiffness K and unstretched length L. The particle is free to move on a circle of radius R which lies on the $z = z_0$ plane. A vertical gravitational force $-mg\mathbf{E}_z$ acts on the particle. If the circle is smooth, show that the resultant force $\mathbf{F}$ acting on the particle has the representation

$$\mathbf{F} = -mg\mathbf{E}_z + N_z\mathbf{E}_z + N_r\mathbf{e}_r - K\left(\sqrt{R^2+z_0^2} - L\right)\frac{R\mathbf{e}_r + z_0\mathbf{E}_z}{\sqrt{R^2+z_0^2}}.$$

4.6. Suppose that the particle in Exercise 4.5 is moving on a rough circle. Show that the resultant force $\mathbf{F}$ acting on the particle now has the representation

$$\mathbf{F} = -mg\mathbf{E}_z + N_z\mathbf{E}_z + N_r\mathbf{e}_r - K\left(\sqrt{R^2+z_0^2} - L\right)\frac{R\mathbf{e}_r + z_0\mathbf{E}_z}{\sqrt{R^2+z_0^2}}$$
$$-\mu_d\sqrt{N_r^2+N_z^2}\frac{\dot{\theta}}{|\dot{\theta}|}\mathbf{e}_\theta.$$

4.7. For the dynamic friction force it is a common *error* to write

$$\mathbf{F}_f = -\mu_d\,\|\mathbf{N}\|\,\frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Give two examples which illustrate that this expression is incorrect when the surface or curve that the particle is moving on is itself in motion.

4.8. A particle of mass m is connected to a fixed point O by a spring of stiffness K and unstretched length L. Show that the spring force has the representations

$$\mathbf{F}_s = -K\left(\sqrt{x^2+y^2+z^2} - L\right)\frac{x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z}{\sqrt{x^2+y^2+z^2}}$$
$$= -K\left(\sqrt{r^2+z^2} - L\right)\frac{r\mathbf{e}_r + z\mathbf{E}_z}{\sqrt{r^2+z^2}}.$$

4.9. For the system considered in Section 4.5, establish the differential equations governing the motion of the particle when the contact between the cone and the particle is smooth. In addition, specialize your results to the case where the spring is replaced by an inextensible cable of length L.

Chapter 5
Power, Work, and Energy

TOPICS

We begin here by discussing the notions of power and work. Subsequently, we make these ideas more precise by defining the mechanical power of a force and, from this, the work done by the force during the motion of a particle. Next, the work-energy theorem $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ is derived from the balance of linear momentum. It is then appropriate to discuss conservative forces, and we spend some added time discussing the potential energies of gravitational and spring forces. With these preliminaries aside, energy conservation is discussed. Finally, some examples are presented that show how all of these ideas are used.

5.1 The Concepts of Work and Power

You may recall from other courses the notion that the work done by a constant force in moving an object through a distance is the product of the force and the distance. This idea has been traced to Coriolis in the 1820s (see [39]).[1] Subsequent works on the notion of work extended the concept to nonconstant forces moving a particle along a curved path at a nonconstant speed. For this more general case, it is convenient to use the concept of mechanical power or the rate at which a force performs work. Energy is defined as the ability to perform work.

In SI units, the units of work are Newton meters (or Joule) and the units of power are Newton meters per second (or Watt). The Scotsman James Watt (1736–1819) played a seminal role in the development of the steam engine, and James P. Joule (1818–1889) was an English physicist who is famed for his discovery of the formula for the heat developed by passing a current through a conductor. The power of an automotive engine is often described in units of horsepower (Hp). Noting that one

[1] Gaspard-Gustav de Coriolis (1792–1843) was a French engineer, mathematician, and scientist. His most famous contribution to science was the discovery of the Coriolis effect.

O.M. O'Reilly, *Engineering Dynamics: A Primer*, DOI 10.1007/978-1-4419-6360-4_5,

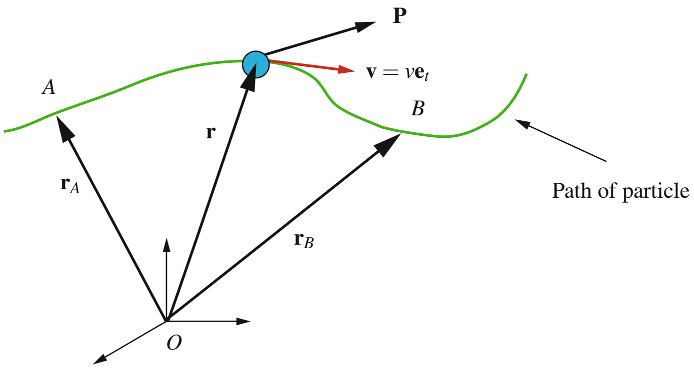

Fig. 5.1 *Schematic of the motion of a particle.*

(metric) Hp ≈ 735.5 Watts, the power rating of the Bugatti Veyron of 1001 metric Hp translates to 736 kiloWatts. Assuming perfect energy conversion, such a car engine has the ability to generate enough electricity to power over seven thousand 100-Watt light bulbs. In nutrition, the unit of work most often used is the calorie ($\approx$ 4200 Joules). Hence, one calorie is sufficient energy to raise 100 kilograms through a distance of ≈ 4.3 meters.

5.2 The Power of a Force

Consider a force $\mathbf{P}$ acting on a particle of mass m. We do not assume that the force is constant or that the particle moves in a straight line. The rate of work done by the force $\mathbf{P}$ on the particle is known as its mechanical power:

$$\text{Mechanical Power of } \mathbf{P} = \mathbf{P}\cdot\mathbf{v}.$$

Here, the particle has a position vector $\mathbf{r}$ and an absolute velocity vector $\mathbf{v} = \dot{\mathbf{r}}$. Consequently, if $\mathbf{P}\cdot\mathbf{v} = 0$ (i.e., $\mathbf{P}$ is normal to $\mathbf{v}$), then the force $\mathbf{P}$ does no work. There are several instances of this occurrence presented in the remainder of this book.

Consider the work done by $\mathbf{P}$ as the particle moves from $\mathbf{r} = \mathbf{r}(t_A) = \mathbf{r}_A$ to $\mathbf{r} = \mathbf{r}(t_B) = \mathbf{r}_B$ (cf. Figure 5.1). At A, the arc-length parameter $s = s_A$, whereas at B, $s = s_B$. In what follows, the vector $\mathbf{e}_t$ is the unit tangent vector to the path of the particle. If the particle is moving on a fixed curve, then this vector is also the unit tangent vector to the fixed curve. On the other hand, if the particle is in motion on a moving curve, then the respective tangent vectors to the curve and the path of the particle will not coincide.[2]

[2] The easiest example that illuminates this point is to consider a particle moving on a horizontal line. The tangent vector is constant, say $\mathbf{E}_x$. However, if the line is moving, say with a velocity

With some algebra, we obtain several equivalent expressions for the work done by **P** by integrating its power with respect to time:

$$\begin{aligned} W_{AB} &= \int_{t_A}^{t_B} \mathbf{P} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{P} \cdot d\mathbf{r} \\ &= \int_{t_A}^{t_B} \mathbf{P} \cdot \frac{ds}{dt} \mathbf{e}_t dt \\ &= \int_{s_A}^{s_B} \mathbf{P} \cdot \mathbf{e}_t ds. \end{aligned}$$

We see from these results, that only the tangential component of **P** does work. In particular, forces that are normal to the path of the particle do no work.

Writing the force **P** and the differential $d\mathbf{r} = \mathbf{v}dt$ with respect to their components in various bases,

$$\begin{aligned} \mathbf{P} &= P_x\mathbf{E}_x + P_y\mathbf{E}_y + P_z\mathbf{E}_z \\ &= P_r\mathbf{e}_r + P_\theta\mathbf{e}_\theta + P_z\mathbf{E}_z \\ &= P_t\mathbf{e}_t + P_n\mathbf{e}_n + P_b\mathbf{e}_b, \\ d\mathbf{r} &= dx\mathbf{E}_x + dy\mathbf{E}_y + dz\mathbf{E}_z \\ &= dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{E}_z = ds\mathbf{e}_t, \end{aligned}$$

we obtain the component forms of the previous results:

$$\begin{aligned} W_{AB} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{P} \cdot d\mathbf{r} &= \int_{\mathbf{r}_A}^{\mathbf{r}_B} P_x dx + P_y dy + P_z dz \\ &= \int_{\mathbf{r}_A}^{\mathbf{r}_B} P_r dr + P_\theta r d\theta + P_z dz \\ &= \int_{s_A}^{s_B} \mathbf{P} \cdot \mathbf{e}_t ds \\ &= \int_{s_A}^{s_B} P_t ds. \end{aligned}$$

These integrals are evaluated along the path of the particle.

$v_c\mathbf{E}_y$, then the velocity vector of the particle is not $v_x\mathbf{E}_x$, rather it is $v_x\mathbf{E}_x + v_c\mathbf{E}_y = \left(\sqrt{v_x^2 + v_c^2}\right)\mathbf{e}_t$, where $\mathbf{e}_t$ is the unit tangent vector to the path of the particle.

5.3 The Work-Energy Theorem

The kinetic energy T of a particle is defined to be

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right),$$

where $\mathbf{v} = v\mathbf{e}_t$. The work-energy theorem relates the change in kinetic energy to the resultant force $\mathbf{F}$ acting on the particle:

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \frac{1}{2}m\dot{\mathbf{v}}\cdot\mathbf{v} + \frac{1}{2}m\mathbf{v}\cdot\dot{\mathbf{v}} = m\dot{\mathbf{v}}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}.$$

In sum, the work-energy theorem is

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v}.$$

In words, the rate of change of the kinetic energy is equal to the mechanical power of the resultant force. You should note that this theorem is a consequence of $\mathbf{F} = m\mathbf{a}$.

We can integrate $\dot{T} = \mathbf{F}\cdot\mathbf{v}$ with respect to time to get a result that you have used previously:

$$\begin{aligned}\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = \int_{t_A}^{t_B}\dot{T}dt &= \int_{t_A}^{t_B}\mathbf{F}\cdot\mathbf{v}dt = \int_{s_A}^{s_B}\mathbf{F}\cdot\mathbf{e}_t ds\\ &= \int_{s_A}^{s_B} m\mathbf{a}\cdot\mathbf{e}_t ds\\ &= m\int_{s_A}^{s_B} a_t ds.\end{aligned}$$

That is,

$$v_B^2 - v_A^2 = \int_{s_A}^{s_B} 2a_t ds.$$

Alternatively, you could derive this result from $a_t = \dot{v} = v dv/ds$.

5.4 Conservative Forces

A force $\mathbf{P}$ is defined to be conservative when

$$\mathbf{P} = -\mathrm{grad}(U) = -\frac{\partial U}{\partial \mathbf{r}}.$$

Here, $U = U(\mathbf{r})$ is the potential energy of the force $\mathbf{P}$, and the negative sign is a historical convention.

The potential energy function U has several representations:

$$U = U(\mathbf{r}) = \bar{U}(s) = \hat{U}(x,y,z) = \tilde{U}(r,\theta,z).$$

Furthermore, this energy is defined modulo an arbitrary additive constant. Here, we always take this constant to be zero. The gradient of U has several representations depending on the coordinate system used:

$$\begin{aligned}\frac{\partial U}{\partial \mathbf{r}} &= \frac{\partial \hat{U}}{\partial x}\mathbf{E}_x + \frac{\partial \hat{U}}{\partial y}\mathbf{E}_y + \frac{\partial \hat{U}}{\partial z}\mathbf{E}_z \\ &= \frac{\partial \tilde{U}}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \tilde{U}}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \tilde{U}}{\partial z}\mathbf{E}_z.\end{aligned}$$

You can also use any of these representations to obtain the results discussed below. However, it is easier to derive most of them without specifying a particular basis or coordinate system.

If a force $\mathbf{P}$ is conservative, then the work done by $\mathbf{P}$ in any motion depends only on the endpoints and not on the path. To see this, we use our earlier results on the work done by $\mathbf{P}$:

$$\begin{aligned}W_{AB} &= \int_{t_A}^{t_B} \dot{W}\,dt = \int_{t_A}^{t_B} \mathbf{P}\cdot\frac{d\mathbf{r}}{dt}dt \\ &= \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{P}\cdot d\mathbf{r} = -\int_{\mathbf{r}_A}^{\mathbf{r}_B} \frac{\partial U}{\partial \mathbf{r}}\cdot d\mathbf{r} \\ &= U(\mathbf{r}_A) - U(\mathbf{r}_B).\end{aligned}$$

Hence, if A and B have the same position vector, then no work is done by $\mathbf{P}$. This leads to the statement that the work done by a conservative force in a closed path of the particle is zero.

It is important to note that if $\mathbf{P}$ is conservative then its mechanical power has a simple expression:

$$\mathbf{P}\cdot\mathbf{v} = -\frac{\partial U}{\partial \mathbf{r}}\cdot\mathbf{v} = -\frac{dU}{dt}.$$

Not all forces are conservative. For example, tension forces in inextensible strings, friction forces $\mathbf{F}_f$, and normal forces $\mathbf{N}$ are not conservative.

5.5 Examples of Conservative Forces

The two main examples of conservative forces one encounters are constant forces $\mathbf{C}$, of which the gravitational force is an example, and spring forces $\mathbf{F}_s$.

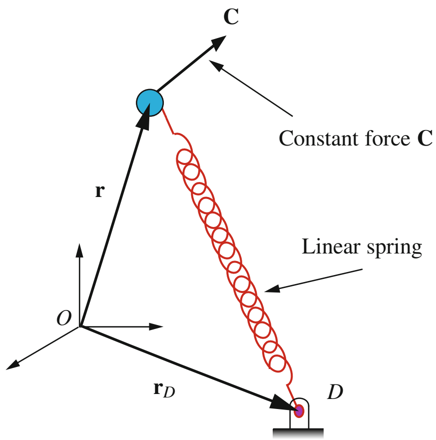

Fig. 5.2 *A particle under the influence of conservative forces* **C** *and* $\mathbf{F}_s$. *The total potential energy of the particle is* $U = U_c + U_s$.

5.5.1 Constant Forces

The conservative nature of the gravitational force $-mg\mathbf{E}_z$ can be seen by defining the potential energy U_c of any constant force $\mathbf{C}$:

$$U_c = -\mathbf{C}\cdot\mathbf{r}.$$

To see that U_c is indeed the potential energy of the force $\mathbf{C}$, we need to show that, for all velocity vectors $\mathbf{v}$,

$$\mathbf{C}\cdot\mathbf{v} = -\frac{\partial U_c}{\partial \mathbf{r}}\cdot\mathbf{v} = -\frac{dU_c}{dt}.$$

Let's do this:

$$\frac{dU_c}{dt} = \frac{d}{dt}\left(-\mathbf{C}\cdot\mathbf{r}\right) = -\dot{\mathbf{C}}\cdot\mathbf{r} - \mathbf{C}\cdot\mathbf{v} = -\mathbf{C}\cdot\mathbf{v}.$$

Based on this result, we then have the following representative constant forces and their potential energies:

$$\begin{aligned}
\mathbf{C} &= -mg\mathbf{E}_y, \qquad U_c = mg\mathbf{E}_y\cdot\mathbf{r},\\
\mathbf{C} &= -mg\mathbf{E}_z, \qquad U_c = mg\mathbf{E}_z\cdot\mathbf{r},\\
\mathbf{C} &= 10\mathbf{E}_x, \qquad U_c = -10\mathbf{E}_x\cdot\mathbf{r}.
\end{aligned}$$

You can easily construct some others. Notice that the gravitational potential energy is "mg" times the "height."

5.5.2 Spring Forces

Consider a linear spring of stiffness K and unstretched length L. As shown in Figure 5.2, one end of the spring is attached to a particle of mass m and the other end is attached to a fixed point D. You should recall that in Section 4.4 of the previous chapter we showed that the force exerted by the spring on the particle is

$$\mathbf{F}_s = -K\left(\|\mathbf{r}-\mathbf{r}_D\| - L\right)\frac{\mathbf{r}-\mathbf{r}_D}{\|\mathbf{r}-\mathbf{r}_D\|}.$$

This force is conservative and has a potential energy

$$U_s = \frac{K}{2}\left(\|\mathbf{r}-\mathbf{r}_D\| - L\right)^2.$$

That is, the potential energy of a linear spring is half the stiffness times the change in length squared.

To show that U_s is indeed the potential energy of $\mathbf{F}_s$, we need to determine its gradient or, equivalently, we need to show that, for all velocity vectors $\mathbf{v}$,

$$\mathbf{F}_s \cdot \mathbf{v} = -\frac{\partial U_s}{\partial \mathbf{r}} \cdot \mathbf{v} = -\frac{dU_s}{dt}.$$

Because $\mathbf{F}_s$ is not constant,

$$\mathbf{F}_s \cdot \mathbf{v} \neq \frac{d}{dt}\left(\mathbf{F}_s \cdot \mathbf{r}\right),$$

and, as a result, this is a difficult result to establish. To show that U_s is the correct potential energy, we first need to establish an intermediate result. Suppose $\mathbf{x}$ is a function of time, then

$$\begin{aligned}
\frac{d\|\mathbf{x}\|}{dt} &= \frac{d}{dt}\left(\sqrt{\mathbf{x}\cdot\mathbf{x}}\right) \\
&= \frac{1}{2\sqrt{\mathbf{x}\cdot\mathbf{x}}}\frac{d}{dt}\left(\mathbf{x}\cdot\mathbf{x}\right) \\
&= \frac{1}{2\sqrt{\mathbf{x}\cdot\mathbf{x}}}\left(\dot{\mathbf{x}}\cdot\mathbf{x} + \mathbf{x}\cdot\dot{\mathbf{x}}\right) \\
&= \frac{\dot{\mathbf{x}}\cdot\mathbf{x}}{\|\mathbf{x}\|}.
\end{aligned}$$

If we let $\mathbf{x} = \mathbf{r} - \mathbf{r}_D$, then we find the result

$$\frac{d\|\mathbf{r}-\mathbf{r}_D\|}{dt} = \frac{(\dot{\mathbf{r}}-\dot{\mathbf{r}}_D)\cdot(\mathbf{r}-\mathbf{r}_D)}{\|\mathbf{r}-\mathbf{r}_D\|} = \frac{\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_D)}{\|\mathbf{r}-\mathbf{r}_D\|}.$$

Let's now differentiate U_s:

$$\begin{aligned}\frac{dU_s}{dt} &= \frac{d}{dt}\left(\frac{K}{2}\left(\|\mathbf{r}-\mathbf{r}_D\|-L\right)^2\right)\\ &= K\left(\|\mathbf{r}-\mathbf{r}_D\|-L\right)\frac{d}{dt}\left(\|\mathbf{r}-\mathbf{r}_D\|-L\right)\\ &= K\left(\|\mathbf{r}-\mathbf{r}_D\|-L\right)\frac{\dot{\mathbf{r}}\cdot(\mathbf{r}-\mathbf{r}_D)}{\|\mathbf{r}-\mathbf{r}_D\|}\\ &= -\mathbf{F}_s\cdot\mathbf{v}.\end{aligned}$$

Hence, U_s is indeed the potential energy of the linear spring force. You should notice that in the proof we used the fact that $\mathbf{v}_D = \mathbf{0}$. Finally, the expression we have obtained for U_s is valid even when $\mathbf{r}_D = \mathbf{0}$, that is, when the point D can be chosen to be the origin.

5.6 Energy Conservation

Consider a particle of mass m that is acted upon by a set of forces: n of these forces are conservative, $\mathbf{F}_1$, $\mathbf{F}_2, \ldots, \mathbf{F}_n$, and the remainder, whose resultant we denote by $\mathbf{F}_{nc}$, are nonconservative. The potential energies of the conservative forces are denoted by U_1, $U_2, \ldots, U_n$:

$$\mathbf{F}_i = -\frac{\partial U_i}{\partial \mathbf{r}},$$

where $i = 1, \ldots, n$. The resultant conservative force acting on the particle is

$$\mathbf{F}_c = \sum_{i=1}^{n}\mathbf{F}_i = -\sum_{i=1}^{n}\frac{\partial U_i}{\partial \mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}},$$

where U, the total potential energy of the conservative forces, is

$$U = \sum_{i=1}^{n} U_i.$$

In summary,

$$\mathbf{F} = \mathbf{F}_{nc} + \mathbf{F}_c = \mathbf{F}_{nc} - \frac{\partial U}{\partial \mathbf{r}}.$$

To establish energy-conservation results, we start with the work-energy theorem:

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v}.$$

For the case at hand,

$$\begin{aligned}\frac{dT}{dt} &= \mathbf{F}\cdot\mathbf{v} = (\mathbf{F}_c + \mathbf{F}_{nc})\cdot\mathbf{v} = \left(-\frac{\partial U}{\partial \mathbf{r}} + \mathbf{F}_{nc}\right)\cdot\mathbf{v} \\ &= -\frac{dU}{dt} + \mathbf{F}_{nc}\cdot\mathbf{v}.\end{aligned}$$

We now define the total energy E of the particle:

$$E = T + U.$$

This energy is the sum of the kinetic and potential energies of the particle. Rearranging the previous equation with the assistance of the definition of E, we find that

$$\frac{dE}{dt} = \mathbf{F}_{nc}\cdot\mathbf{v}.$$

This equation can be viewed as an alternative form of the work-energy theorem.

If the nonconservative forces do no work during a motion of the particle, that is, $\mathbf{F}_{nc}\cdot\mathbf{v} = \mathbf{0}$, then the total energy E of the particle is conserved:[3]

$$\frac{dE}{dt} = 0.$$

This implies that E is a constant E_0 during the motion of the particle:[4]

$$E = T + U = \frac{1}{2}mv^2 + U(\mathbf{r}) = E_0.$$

During an energy-conserving motion, there is a transfer between the kinetic and potential energies of a particle.

In problems, one uses energy conservation $\dot{E} = 0$ to solve for one unknown. For example, suppose one is given an initial speed v_0 and location $\mathbf{r}_0$ of a particle at some instant during an energy-conserving motion. One can use energy conservation to determine the speed v at another location $\mathbf{r}$ during this motion:

$$v^2 = \frac{2}{m}\left(U(\mathbf{r}_0) - U(\mathbf{r})\right) + v_0^2.$$

In light of our earlier discussion in Section 5.3 of the identity $a_t = dv/dt = vdv/ds$, you should notice that

$$\frac{1}{m}\left(U(\mathbf{r}_0) - U(\mathbf{r})\right) = \int_{s_0}^{s} a_t(u)du = \frac{1}{m}\int_{t_0}^{t} \mathbf{F}\cdot\mathbf{v}d\tau.$$

[3] This is a classical result that was known, although not in the form written here, to the Dutch scientist Christiaan Huygens (1629–1695) and the German scientist Gottfried Wilhelm Leibniz (1646–1716). These men were contemporaries of Isaac Newton (1643–1727).

[4] A common error is to assume that $\dot{E} = 0$ implies that $E = 0$. It does not.

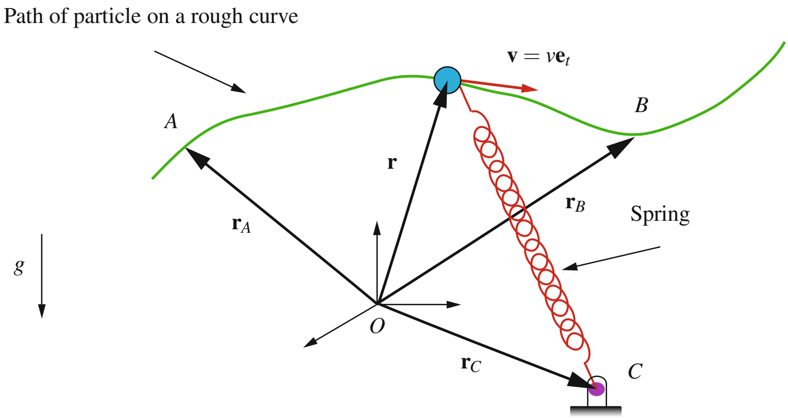

Fig. 5.3 *A particle in motion on a fixed space curve. The total potential energy of the particle is the sum of the gravitational and spring potential energies.*

That is, if the only forces that do work are conservative, then the existence of the potential energies makes integrating the power of **F** trivial.

5.7 A Particle Moving on a Rough Curve

Consider a particle of mass m that is moving on the rough space curve shown in Figure 5.3. A gravitational force $-mg\mathbf{E}_z$ acts on the particle. In addition, a linear spring of stiffness K and unstretched length L is attached to the particle and a fixed point C.

We wish to determine the work done by the friction force on the particle as it moves from $\mathbf{r} = \mathbf{r}(t_A) = \mathbf{r}_A$ to $\mathbf{r} = \mathbf{r}(t_B) = \mathbf{r}_B$. Furthermore, we ask the question, if the curve were smooth, then given $\mathbf{v}(t_A)$, what would $\mathbf{v}(t_B)$ be?

5.7.1 Kinematics

Here, we use the Serret-Frenet triads, and we tacitly assume that these vectors can be calculated for the curve at hand. We then have the usual results

$$\mathbf{v} = v\mathbf{e}_t, \qquad \mathbf{a} = \dot{v}\mathbf{e}_t + \kappa v^2\mathbf{e}_n, \qquad T = \frac{1}{2}mv^2.$$

5.7.2 Forces

We leave the free-body diagram as an exercise and record that the resultant force $\mathbf{F}$ acting on the particle is

$$\mathbf{F} = \mathbf{F}_f + N_n\mathbf{e}_n + N_b\mathbf{e}_b - mg\mathbf{E}_z - K\left(\|\mathbf{r}-\mathbf{r}_C\| - L\right)\frac{\mathbf{r}-\mathbf{r}_C}{\|\mathbf{r}-\mathbf{r}_C\|}.$$

If the friction is dynamic Coulomb friction, then

$$\mathbf{F}_f = -\mu_d \|N_n\mathbf{e}_n + N_b\mathbf{e}_b\| \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

On the other hand, if the particle is stationary, then the friction is static Coulomb friction.

5.7.3 Work Done by Friction

We are now in a position to start from the work-energy theorem and establish how the rate of change of total energy is related to the power of the friction force:

$$\begin{aligned}\frac{dT}{dt} &= \mathbf{F}\cdot\mathbf{v}\\ &= \mathbf{F}_f\cdot\mathbf{v} + N_n\mathbf{e}_n\cdot\mathbf{v} + N_b\mathbf{e}_b\cdot\mathbf{v} - mg\mathbf{E}_z\cdot\mathbf{v} - K\left(\|\mathbf{r}-\mathbf{r}_C\| - L\right)\frac{\mathbf{r}-\mathbf{r}_C}{\|\mathbf{r}-\mathbf{r}_C\|}\cdot\mathbf{v}.\end{aligned}$$

However, the normal forces are perpendicular to the velocity vector, and the spring and gravitational forces are conservative:

$$\begin{aligned}N_n\mathbf{e}_n\cdot\mathbf{v} &= 0,\\ N_b\mathbf{e}_b\cdot\mathbf{v} &= 0,\\ -mg\mathbf{E}_z\cdot\mathbf{v} &= -\frac{d}{dt}\left(mg\mathbf{E}_z\cdot\mathbf{r}\right),\\ -K\left(\|\mathbf{r}-\mathbf{r}_C\| - L\right)\frac{\mathbf{r}-\mathbf{r}_C}{\|\mathbf{r}-\mathbf{r}_C\|}\cdot\mathbf{v} &= -\frac{d}{dt}\left(\frac{K}{2}\left(\|\mathbf{r}-\mathbf{r}_C\| - L\right)^2\right).\end{aligned}$$

As a result,

$$\frac{dE}{dt} = \mathbf{F}_f\cdot\mathbf{v},$$

where the total energy of the particle E is

$$E = T + \frac{K}{2}\left(\|\mathbf{r}-\mathbf{r}_C\| - L\right)^2 + mg\mathbf{E}_z\cdot\mathbf{r}.$$

You could also have arrived at this result by starting with the alternative form of the work-energy theorem, $\dot{E} = \mathbf{F}_{nc} \cdot \mathbf{v}$, and noting that the normal forces are perpendicular to the velocity of the particle and hence are workless.

The work done by the friction force can be found by integrating the above equation for the time rate of change of the total energy E:

$$\begin{aligned}\int_{t_A}^{t_B} \mathbf{F}_f \cdot \mathbf{v} dt &= E_B - E_A \\ &= \frac{1}{2} m \left(v_B^2 - v_A^2\right) + mg\mathbf{E}_z \cdot (\mathbf{r}_B - \mathbf{r}_A) \\ &\quad + \frac{K}{2} \left((\|\mathbf{r}_B - \mathbf{r}_C\| - L)^2 - (\|\mathbf{r}_A - \mathbf{r}_C\| - L)^2 \right).\end{aligned}$$

Clearly, if you know $\mathbf{r}_A$, $\mathbf{r}_B$, v_A, and v_B, then you don't need to directly integrate $\mathbf{F}_f \cdot \mathbf{v}$ to determine the work done by the friction force. Furthermore, given the above information, one doesn't need to calculate the Serret-Frenet triads for each point along the curve.

It is interesting to note that if the friction were of the dynamic Coulomb type, then

$$\begin{aligned}\frac{dE}{dt} &= \mathbf{F}_f \cdot \mathbf{v} = -\mu_d \|N_n \mathbf{e}_n + N_b \mathbf{e}_b\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{v} \\ &= -\mu_d \|N_n \mathbf{e}_n + N_b \mathbf{e}_b\| \|\mathbf{v}\| < 0.\end{aligned}$$

In other words, such a friction force will dissipate energy, as expected.

5.7.4 *The Smooth Curve*

For a smooth curve, $\mathbf{F}_f = \mathbf{0}$ and we have energy conservation: $\dot{E} = 0$. In this case, given $\mathbf{r}_A$, $\mathbf{r}_B$, and v_A, one can calculate v_B by equating E_A to E_B:

$$v_B^2 = v_A^2 + 2g\mathbf{E}_z \cdot (\mathbf{r}_A - \mathbf{r}_B) + \frac{K}{m} \left((\|\mathbf{r}_A - \mathbf{r}_C\| - L)^2 - (\|\mathbf{r}_B - \mathbf{r}_C\| - L)^2 \right).$$

This is the main use of energy conservation in the problems discussed in engineering dynamics courses. Notice that because the only forces that do work on the particle are conservative, the velocity at B does not depend on the path between A and B.

A specific example of this case was discussed in Section 3.5. In this example, the spring force was absent, and it was shown there, without using the work-energy theorem directly, that

$$v^2(x) = v^2(x_0) + \int_{x_0}^{x} 4ugdu = v^2(x_0) + 2g\left(x^2 - x_0^2\right).$$

Our earlier comment in Section 3.5.4 of Chapter 3 that this was an energy conservation result should now be obvious to you.

5.8 Further Examples of Energy Conservation

We can use the results of the previous problem for many others after performing some minor modifications.

5.8.1 A Particle in a Conservative Force Field

Consider a particle moving in space, as opposed to along a fixed curve. The particle is assumed to be under the sole influences of a gravitational force and a spring force:

$$\mathbf{F} = -mg\mathbf{E}_z - K\left(\|\mathbf{r} - \mathbf{r}_C\| - L\right)\frac{\mathbf{r} - \mathbf{r}_C}{\|\mathbf{r} - \mathbf{r}_C\|}.$$

By setting $\mathbf{F}_f = \mathbf{0}$ and $\mathbf{N} = \mathbf{0}$ in the previous example, one finds that the total energy E of the particle is conserved, where

$$E = T + \frac{K}{2}\left(\|\mathbf{r} - \mathbf{r}_C\| - L\right)^2 + mg\mathbf{E}_z \cdot \mathbf{r}.$$

The Serret-Frenet triad for this example pertains to the path of the particle as opposed to the prescribed fixed curve, and it is impossible to explicitly determine this triad without first solving $\mathbf{F} = m\mathbf{a}$ for the motion $\mathbf{r}(t)$ of the particle.

5.8.2 A Particle on a Fixed Smooth Surface

If a particle is moving on any fixed smooth surface under conservative gravitational and spring forces, then

$$\mathbf{F} = -mg\mathbf{E}_z - K\left(\|\mathbf{r} - \mathbf{r}_C\| - L\right)\frac{\mathbf{r} - \mathbf{r}_C}{\|\mathbf{r} - \mathbf{r}_C\|} + \mathbf{N}.$$

Here, $\mathbf{N}$ is the normal (or reaction) force exerted by the surface on the particle. However, because this force is perpendicular to the velocity vector of the particle, $\mathbf{N} \cdot \mathbf{v} = 0$. Consequently, the total energy is again conserved, where

$$E = T + U, \qquad U = \frac{K}{2}\left(\|\mathbf{r} - \mathbf{r}_C\| - L\right)^2 + mg\mathbf{E}_z \cdot \mathbf{r}.$$

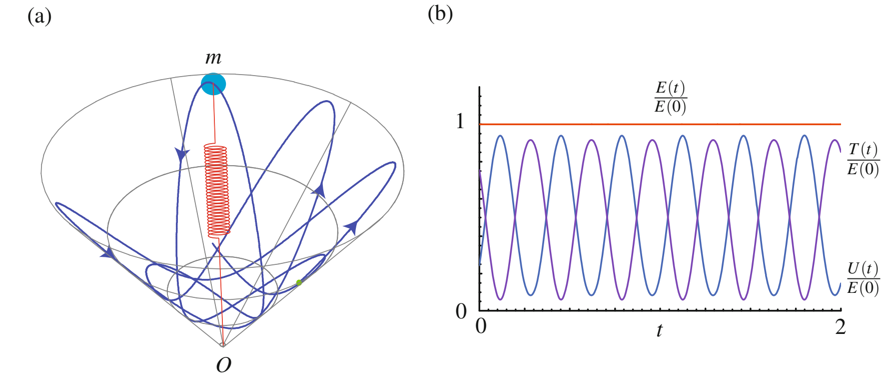

Fig. 5.4 *Conservation of energy for a particle moving on a smooth cone under the action of gravity and a linear spring force: (a) a motion of the particle and (b) the corresponding total energy $E(t)$, kinetic energy $T(t)$, and potential energy $U(t)$ for the particle.*

We have already seen two examples of energy conservation: the pair of motions of the particle shown in Figure 4.9 are such that the total energy E of the particle remains equal to its initial value. The conservation for the motion shown in Figure 4.9(a) is illustrated in Figure 5.4. You should notice how the potential and kinetic energies of the particle vary in time and their sum is constant.

5.8.3 The Planar Pendulum

We described the planar pendulum in Section 2.4. For this example, one can show using the work-energy theorem that the total energy E of the particle is conserved. To begin,

$$\begin{aligned}\frac{dT}{dt} &= \mathbf{F}\cdot\mathbf{v}\\ &= N_r\mathbf{e}_r\cdot\mathbf{v} - mg\mathbf{E}_y\cdot\mathbf{v} + N\mathbf{E}_z\cdot\mathbf{v}\\ &= -\frac{d}{dt}\left(mg\mathbf{E}_y\cdot\mathbf{r}\right),\end{aligned}$$

where we have changed notation and defined $N_r\mathbf{e}_r$ as the tension force in the string/rod. This force and the normal force $N\mathbf{E}_z$ are perpendicular to the velocity vector and, as a result, are workless. It now follows that the total energy E of the particle is conserved:

$$\frac{d}{dt}\left(E = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + mg\mathbf{E}_y\cdot\mathbf{r}\right) = 0.$$

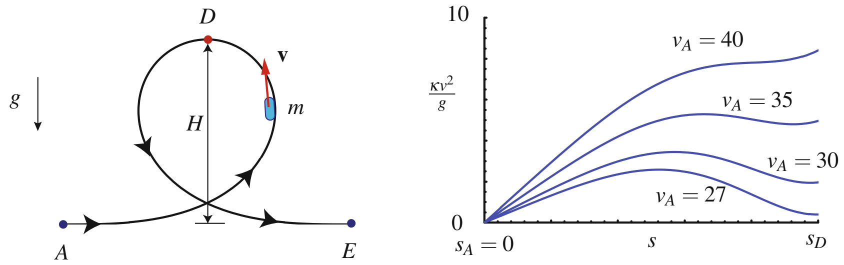

Fig. 5.5 *Schematic of a particle of mass m moving on a smooth clothoid loop, and the normalized acceleration $\kappa v^2/g$ experienced by the particle, for a range of initial velocities v_A, as it moves from the bottom A to the top D of the loop.*

An example of this conservation can be seen in the pendulum motion displayed in Figure 2.1.

5.8.4 The Roller Coaster

Consider a loop-the-loop design for a section of a roller coaster that is shown in Figure 5.5. The design features the clothoid loop that was discussed earlier in Chapter 3. We assume that the loop is smooth and model the roller coaster cart as a particle of mass m. Our interest lies in examining the normal acceleration κv^2 of the cart as it moves along the smooth loop.

The problem at hand is a special case of the situation discussed earlier of a particle moving on a smooth curve under the influence of conservative forces (see Section 5.7.4). Thus, we have the energy conservation result

$$v^2 = v_A^2 + 2g\,(y_A - y)\,.$$

We take the point A as shown in Figure 5.5 and set the arc-length parameter $s_A = 0$ and position vector $\mathbf{r}_A = \mathbf{0}$. It follows readily that the normal acceleration divided by the gravitational acceleration (for a point on the track whose vertical ordinate is y) is given by

$$\frac{\kappa v^2}{g} = \kappa\left(\frac{v_A^2}{g} - 2y\right).$$

Using the results of Exercise 3.11 (see page 51), we note that $\kappa = s/a^2$ where a is a constant and we also note the expression for y. By picking sufficiently large values of v_A we can determine how $\kappa v^2/g$ changes as the particle moves along the loop.[5]

[5] It can be shown for a loop with $H = 35$ meters that $v_A > \sqrt{2gH} \approx 26.21$ meters per second in order for the particle to reach the top D of the loop. By way of background, 27 meters per second is approximately 60 mph, 35 meters is approximately 115 feet, and a value of $H = 35$ corresponds to a value of the clothoid parameter $a = 27.657$.

A range of numerical results is shown in Figure 5.5. For these results, the height H of the loop is 35 meters, and we consider a range of values of v_A from 27 meters per second to 40 meters per second. From the figure, you might notice the high g-force values experienced by the particle and the fact that the maximum normal acceleration does not always occur when the particle is at the apex D of the loop. For roller coaster designs the duration and magnitude of the normal accelerations are relevant to passenger safety and must be considered in the design.[6]

5.9 Summary

The first concept introduced in this chapter was the mechanical power of a force $\mathbf{P}$ acting on a particle whose absolute velocity vector is $\mathbf{v}$: $\mathbf{P}\cdot\mathbf{v}$. The work done by a force $\mathbf{P}$ in a interval of time $[t_A, t_B]$ is the integral of its power with respect to time:

$$W_{AB} = \int_{t_A}^{t_B} \mathbf{P}\cdot\mathbf{v}dt.$$

Depending on the coordinate system used, there are numerous representations of this integral. You should notice that in order to evaluate the integral it is necessary to know the path of the particle. If, for all possible paths, $W_{AB} = 0$, then the force $\mathbf{P}$ does no work and its power must be zero. In other words, $\mathbf{P}$ must be normal to $\mathbf{v}$.

An important class of forces was then discussed: conservative forces. A force $\mathbf{P}$ is conservative if one can find a potential energy function $U = U(\mathbf{r})$ such that, for all possible motions,

$$\mathbf{P} = -\frac{\partial U}{\partial \mathbf{r}},$$

or, equivalently,

$$\dot{U} = -\mathbf{P}\cdot\mathbf{v}.$$

Because a conservative force is the gradient of a scalar function $U = U(\mathbf{r})$, the work done by this class of forces is independent of the path of the particle. In Section 5.5 of this chapter, it was shown that a spring force $\mathbf{F}_s$ and a constant force $\mathbf{C}$ are conservative:

$$\mathbf{F}_s = -K\left(\|\mathbf{r}-\mathbf{r}_D\| - L\right)\frac{\mathbf{r}-\mathbf{r}_D}{\|\mathbf{r}-\mathbf{r}_D\|} = -\frac{\partial U_s}{\partial \mathbf{r}},$$
$$\mathbf{C} = -\frac{\partial U_c}{\partial \mathbf{r}},$$

where

$$U_s = \frac{K}{2}\left(\|\mathbf{r}-\mathbf{r}_D\| - L\right)^2, \qquad U_c = -\mathbf{C}\cdot\mathbf{r}.$$

[6] For a discussion on the relationships between brain trauma and g-forces in roller coaster rides, we refer the reader to [72].

Not all forces are conservative. In particular, friction and normal forces are nonconservative.

Using the notion of mechanical power, two versions of the work-energy theorem were established:

$$\dot{T} = \mathbf{F} \cdot \mathbf{v}, \qquad \dot{E} = \mathbf{F}_{nc} \cdot \mathbf{v}.$$

Here, $T = \frac{m}{2}\mathbf{v} \cdot \mathbf{v}$ is the kinetic energy of the particle, $E = T + U$ is the total energy of the particle, U is the sum of the potential energies of all the conservative forces acting on the particle, and $\mathbf{F}_{nc}$ is the resultant nonconservative force acting on the particle. You should notice that $\mathbf{F}_{nc} = \mathbf{F} + \partial U / \partial \mathbf{r}$.

The remainder of the chapter was concerned with using the work-energy theorem to examine the work done by friction forces and providing examples of systems where the total energy E was conserved. For all of the examples of energy conservation, the work-energy theorem was used to show that $\mathbf{F}_{nc} \cdot \mathbf{v} = 0$ and, consequently, E must be conserved. As illustrated in the examples of energy conservation, when E is conserved then certain information on the speed of the particle as a function of position can be determined without explicitly integrating the differential equations governing the motion of the particle.

5.10 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

5.1. Give examples to illustrate the following statement: "Every constant force is conservative, but not all conservative forces are constant."

5.2. Starting from the definition of the kinetic energy T, establish the work-energy theorem $\dot{T} = \mathbf{F} \cdot \mathbf{v}$. Then, using this result, derive the alternative form of the work-energy theorem: $\dot{E} = \mathbf{F}_{nc} \cdot \mathbf{v}$.

5.3. Give three examples of particle problems where the total energy E is conserved.

5.4. A particle is moving on a smooth horizontal plane. A gravitational force $-mg\mathbf{E}_z$ acts on the particle. If the plane is given a vertical motion, then why does the normal force acting on the particle perform work? Using this example, show that the normal force is not a conservative force.

5.5. Give three examples of particle problems where the total energy E is not conserved.

5.6. A particle is free to move on a smooth plane $z = 0$. It is attached to a fixed point O by a linear spring of stiffness K and unstretched length L. A gravitational force $-mg\mathbf{E}_z$ acts on the particle. Starting from the work-energy theorem, prove that

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{2}(r - L)^2$$

is conserved.

5.7. Consider the same system discussed in Exercise 5.6, but in this case assume that the surface is rough. Show that

$$\dot{E} = -\mu_d mg\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}.$$

5.8. For any vector **b** show that

$$\frac{d}{dt}\|\mathbf{b}\| = \frac{\mathbf{b}\cdot\dot{\mathbf{b}}}{\|\mathbf{b}\|}.$$

Using this result, show that if the magnitude of **b** is constant, then any change in **b** must be normal to **b**.

5.9. With the assistance of the identity established in Exercise 5.8, show that Newton's gravitational force,

$$\mathbf{F}_N = -\frac{GMm}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|},$$

is conservative with a potential energy

$$U_N = -\frac{GMm}{\|\mathbf{r}\|}.$$

Here, G, M, and m are constants.

Part II
Dynamics of a System of Particles

Chapter 6
Momenta, Impulses, and Collisions

TOPICS

As a prelude to the discussion of a system of particles, the linear and angular momenta of a single particle are introduced in this chapter. In particular, conditions for the conservation of these kinematical quantities are established. This is followed by a discussion of impact problems where particles are used as models for the impacting bodies.

6.1 Linear Momentum and Its Conservation

Consider a particle of mass m moving in space. As usual, the position vector of the particle relative to a fixed origin is denoted by $\mathbf{r}$. We recall that the linear momentum $\mathbf{G}$ of the particle is defined to be

$$\mathbf{G} = m\mathbf{v} = m\dot{\mathbf{r}}.$$

6.1.1 Linear Impulse and Linear Momentum

A more primitive form of the balance of linear momentum $\mathbf{F} = m\mathbf{a}$ is its integral form:

$$\mathbf{G}(t_1) - \mathbf{G}(t_0) = \int_{t_0}^{t_1} \mathbf{F} dt.$$

This equation is assumed to hold for all intervals of time and hence for all times t_0 and t_1. The time integral of a force is known as its *linear impulse*.

The reason the integral form is more primitive than the equation $\mathbf{F} = m\mathbf{a}$ is that it does not assume that $\mathbf{v}$ can always be differentiated to determine $\mathbf{a}$. We use the integral form of the balance of linear momentum in our forthcoming discussion of impact.

6.1.2 Conservation of Linear Momentum

The conservation of some component of linear momentum of the particle is an important feature of many problems. Suppose that the component of $\mathbf{G}$ in the direction of a *given* vector $\mathbf{c}$ is conserved:

$$\frac{d}{dt}(\mathbf{G}\cdot\mathbf{c}) = 0.$$

To examine the conditions under which this arises, we expand the time derivative on the right-hand side of the above equation to find that

$$\frac{d}{dt}(\mathbf{G}\cdot\mathbf{c}) = \dot{\mathbf{G}}\cdot\mathbf{c} + \mathbf{G}\cdot\dot{\mathbf{c}}.$$

Consequently, given a vector $\mathbf{c}$,

$$\mathbf{G}\cdot\mathbf{c} \text{ is conserved if, and only if, } \mathbf{F}\cdot\mathbf{c} + \mathbf{G}\cdot\dot{\mathbf{c}} = 0.$$

A special case of this result arises when $\mathbf{c}$ is constant. In this case, the condition for the conservation of the linear momentum in the direction of $\mathbf{c}$ is none other than $\mathbf{F}\cdot\mathbf{c} = 0$. That is, there is no force in this direction.

In general, the most difficult aspect of using the conservation of linear momentum is to find appropriate directions $\mathbf{c}$. This is an art.

6.1.3 Examples

You have already seen several examples of linear momentum conservation. For instance, consider a particle moving in a gravitational field. Here, $\mathbf{F} = -mg\mathbf{E}_y$. As a result, $\mathbf{F}\cdot\mathbf{E}_z = 0$ and $\mathbf{F}\cdot\mathbf{E}_x = 0$. The linear momenta of the particle and, as a result, its velocity in the directions $\mathbf{E}_x$ and $\mathbf{E}_z$ are conserved. It is perhaps instructive to note that the linear momentum of the particle in the $\mathbf{E}_y$ direction is not conserved.

6.2 Angular Momentum and Its Conservation

As shown in Figure 6.1, let $\mathbf{r}$ be the position vector of a particle relative to a fixed point O, and let $\mathbf{v}$ be the absolute velocity vector of the particle. Then, the angular

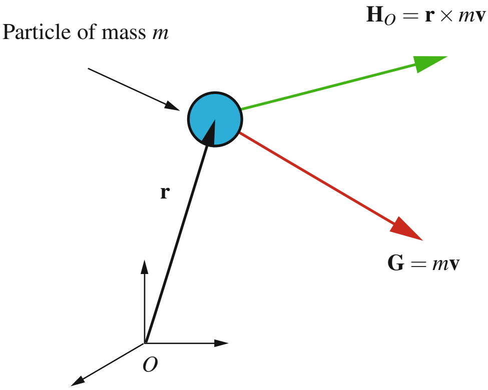

Fig. 6.1 *Some kinematical quantities pertaining to a particle of mass m.*

momentum of the particle relative to O is denoted by $\mathbf{H}_O$ and defined as

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times \mathbf{G}.$$

In Cartesian coordinates, $\mathbf{H}_O$ has the representation

$$\begin{aligned}\mathbf{H}_O &= \mathbf{r} \times m\mathbf{v} \\ &= \det \begin{bmatrix} \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{bmatrix} \\ &= m(y\dot{z} - z\dot{y})\mathbf{E}_x + m(z\dot{x} - x\dot{z})\mathbf{E}_y + m(x\dot{y} - y\dot{x})\mathbf{E}_z.\end{aligned}$$

In cylindrical polar coordinates, $\mathbf{H}_O$ has the representation

$$\begin{aligned}\mathbf{H}_O &= \mathbf{r} \times m\mathbf{v} \\ &= \det \begin{bmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{E}_z \\ r & 0 & z \\ m\dot{r} & mr\dot{\theta} & m\dot{z} \end{bmatrix} \\ &= -mzr\dot{\theta}\mathbf{e}_r + m(z\dot{r} - r\dot{z})\mathbf{e}_\theta + mr^2\dot{\theta}\mathbf{E}_z.\end{aligned}$$

When the motion of the particle is planar,

$$\mathbf{r} = r\mathbf{e}_r, \qquad \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$

and two of the previous representations for $\mathbf{H}_O$ simplify to

$$\mathbf{H}_O = m\,(x\dot{y} - y\dot{x})\,\mathbf{E}_z = mr^2\dot{\theta}\mathbf{E}_z.$$

6.2.1 Angular Momentum Theorem

To determine how the angular momentum changes with time, we invoke the balance of linear momentum and compute an expression for $\dot{\mathbf{H}}_O$:

$$\frac{d\mathbf{H}_O}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{r} \times \mathbf{F}.$$

The final result is known as the angular momentum theorem:

$$\frac{d\mathbf{H}_O}{dt} = \mathbf{r} \times \mathbf{F}.$$

As opposed to our later developments with rigid bodies, a balance of angular momentum $\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F}$ is not an independent postulate. It arises as a consequence of the balance of linear momentum.

6.2.2 Conservation of Angular Momentum

The conservation of some component of angular momentum of the particle is an important feature of many problems. Suppose that the component of $\mathbf{H}_O$ in the direction of a *given* vector $\mathbf{c}$ is conserved:

$$\frac{d}{dt}(\mathbf{H}_O \cdot \mathbf{c}) = 0.$$

To examine the conditions under which this arises, we expand the time derivative on the left-hand side of the above equation and invoke the angular momentum theorem to find that

$$\begin{aligned}\frac{d}{dt}(\mathbf{H}_O \cdot \mathbf{c}) &= \dot{\mathbf{H}}_O \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} \\ &= (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}}.\end{aligned}$$

Consequently, for a given vector $\mathbf{c}$,

$$\mathbf{H}_O \cdot \mathbf{c} \text{ is conserved if, and only if, } (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = 0.$$

A special case of this result arises when $\mathbf{c}$ is constant. Then, the condition for the conservation of the angular momentum in the direction of $\mathbf{c}$ is none other than $(\mathbf{r} \times \mathbf{F}) \cdot \mathbf{c} = 0$.

In an undergraduate dynamics course, problems where angular momentum is conserved can usually be set up so that $\mathbf{c} = \mathbf{E}_z$. We shortly examine such an example.

6.2.3 Central Force Problems

A central force problem is one where $\mathbf{F}$ is parallel to $\mathbf{r}$. The angular momentum theorem in this case implies that $\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F} = \mathbf{0}$. It follows that $\mathbf{H}_O$ is conserved. This conservation implies several interesting results, which we now discuss.

Because $\mathbf{H}_O$ is conserved, $\mathbf{H}_O = h\mathbf{h}$, where h and $\mathbf{h}$ are constant. We can choose $\mathbf{h}$ to be a unit vector. Because $\mathbf{r} \times m\mathbf{v}$ is constant, the vectors $\mathbf{r}$ and $\mathbf{v}$ form a plane with a constant unit normal vector $\mathbf{h}$. This plane passes through the origin O and is fixed. Given a set of initial conditions $\mathbf{r}(t_0)$ and $\mathbf{v}(t_0)$, we can choose a cylindrical polar coordinate system such that $\mathbf{E}_z = \mathbf{h}$, $\mathbf{r} = r\mathbf{e}_r$, and $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$. To do this, it suffices to choose $\mathbf{E}_z$ so that

$$\mathbf{H}_O = h\mathbf{E}_z = \mathbf{r}(t_0) \times m\mathbf{v}(t_0).$$

This simplifies the problem of determining the motion of the particle considerably. Furthermore, $h = mr^2\dot{\theta}$ is constant during the motion of the particle.

6.2.4 Kepler's Problem

The most famous example of a central force problem, and angular momentum conservation, was solved by Newton. In seeking to develop a model for planetary motion which would explain Kepler's laws[1] and astronomical observations, he postulated a model for the resultant force $\mathbf{F}$ exerted on a planet of mass m by a fixed planet of mass M. The force $\mathbf{F}$ that Newton postulated was conservative:

$$\mathbf{F} = -\frac{GmM}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{\partial U}{\partial \mathbf{r}}, \qquad U = -\frac{GmM}{\|\mathbf{r}\|}.$$

Here, the fixed planet is located at the origin O and G is the universal gravitational constant:

$$G \approx 6.673 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}.$$

By way of additional background, the mass of the Sun is $\approx 2 \times 10^{30}$ kilograms whereas the mass of the Earth is $\approx 5.98 \times 10^{24}$ kilograms. Clearly, the force that Newton postulated is an example of a central force. Thus, we can use the previous results on central force problems here.

Using the balance of linear momentum for the planet of mass m, we find two ordinary differential equations:

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{GMm}{r^2}, \qquad mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0.$$

[1] Johannes Kepler (1571–1630) was a German astronomer and physicist who, based on observations of the orbits of certain planets, proposed three laws to explain planetary motion. His three laws are the most famous of his many scientific contributions.

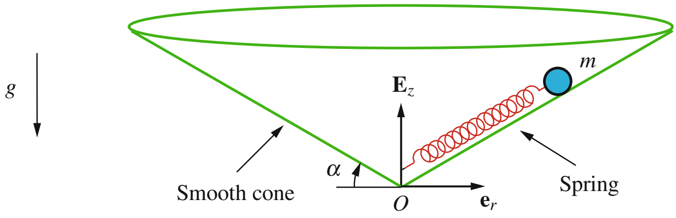

Fig. 6.2 *A particle moving on a smooth cone.*

The solution of these equations is facilitated by the use of two conserved quantities, the total energy E and the magnitude of the angular momentum h:

$$E = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GmM}{r},$$
$$h = \mathbf{H}_O \cdot \mathbf{E}_z = mr^2\dot{\theta}.$$

However, we do not pursue this matter any further here.[2]

6.2.5 A Particle on a Smooth Cone

As shown in Figure 6.2, we return to an example discussed in Section 4.5 of Chapter 4. Here, however, the surface of the cone is assumed to be smooth. We wish to show that $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved for the particle.[3]

Let us first recall some kinematical results from Section 4.5 of Chapter 4:

$$\mathbf{r} = r\mathbf{e}_r + r\tan(\alpha)\mathbf{E}_z,$$
$$\mathbf{v} = \dot{r}\left(\mathbf{e}_r + \tan(\alpha)\mathbf{E}_z\right) + r\dot{\theta}\mathbf{e}_\theta.$$

A simple calculation shows that the angular momentum of the particle is

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} = \frac{mr^2\dot{\theta}}{\cos(\alpha)}\left(\cos(\alpha)\mathbf{E}_z - \sin(\alpha)\mathbf{e}_r\right).$$

---

[2] For further treatments of these equations, see, for example, Arnol'd [2] and Moulton [50]. Discussions of this central force problem can also be found in every undergraduate dynamics text, for example, in Section 13, Chapter 3 of Meriam and Kraige [48] and Section 15, Chapter 5 of Riley and Sturges [63]. You should notice that these texts assume that the motion of the particle is planar.

[3] We leave it as an exercise to show that the total energy and angular momentum $\mathbf{H}_O \cdot \mathbf{E}_z$ are also conserved when the spring is replaced by an inextensible string.

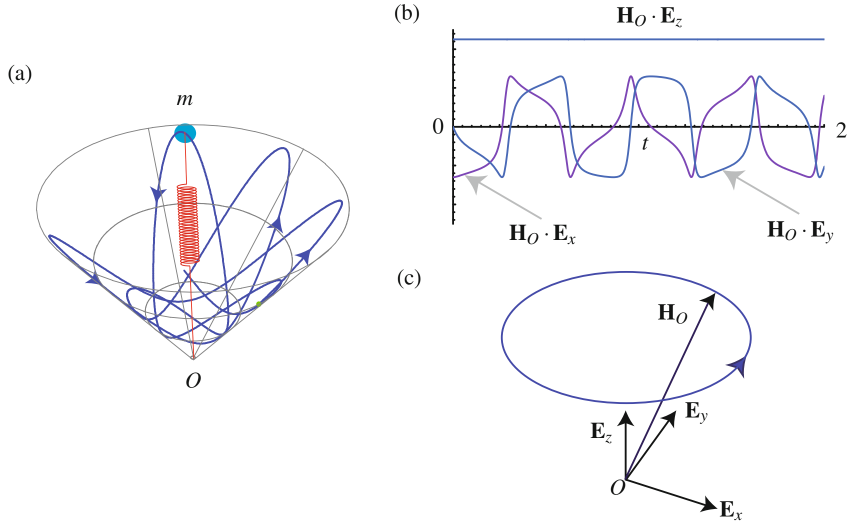

Fig. 6.3 *Conservation of* $\mathbf{H}_O \cdot \mathbf{E}_z$ *for a particle moving on a smooth cone under the action of gravity and a linear spring force: (a) a motion of the particle, (b) the corresponding angular momentum components* $\mathbf{H}_O \cdot \mathbf{E}_x$, $\mathbf{H}_O \cdot \mathbf{E}_y$, *and* $\mathbf{H}_O \cdot \mathbf{E}_z$ *for the particle, and (c) the angular momentum vector* $\mathbf{H}_O$.

We also note that the resultant force acting on the particle is composed of a normal force, a gravitational force, and a spring force:

$$\mathbf{F} = N\mathbf{n} - mg\mathbf{E}_z - K\left(\|\mathbf{r}\| - L\right)\frac{\mathbf{r}}{\|\mathbf{r}\|}.$$

The moment of the resultant force is in the $\mathbf{e}_\theta$ direction:

$$\begin{aligned}\mathbf{r} \times \mathbf{F} &= \mathbf{r} \times N\mathbf{n} - \mathbf{r} \times mg\mathbf{E}_z + \mathbf{r} \times \left(-K\left(\|\mathbf{r}\| - L\right)\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) \\ &= \left(mgr - \frac{Nr}{\cos(\alpha)}\right)\mathbf{e}_\theta.\end{aligned}$$

Consequently, $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved:

$$mr^2\dot{\theta} = \text{constant}.$$

You should notice that during the motion of the particle it is impossible for $\dot{\theta}$ to change sign. Indeed, two examples of this phenomenon were shown earlier in Figure 4.9. To further illustrate the conservation of $\mathbf{H}_O \cdot \mathbf{E}_z$, we plot this component for a motion of the particle in Figure 6.3(b). As shown in Figure 6.3(c), if we construct the angular momentum vector $\mathbf{H}_O$ for the motion of the particle, we would find that, as it evolves in time, it traces out a cone.

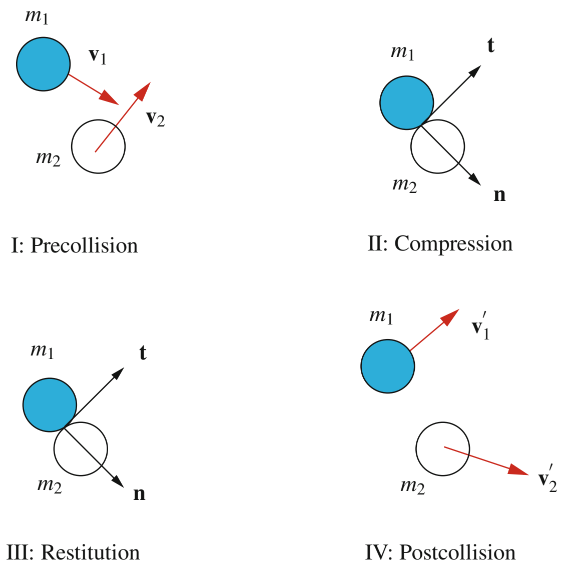

Fig. 6.4 *The four stages of a collision.*

6.3 Collision of Particles

The collision of two bodies involves substantial deformations during the impact and may induce permanent deformations. Ignoring the rotational motion of the bodies, the simplest model to determine the postcollision velocities of the bodies is to use a mass particle to model each individual body. However, mass particles cannot deform, and so one needs to introduce some (seemingly ad hoc) parameter to account for this feature. The parameter commonly used is the coefficient of restitution e.

The theory we present here is often referred to as frictionless, oblique, central impact of two particles. Other theories are available that account for friction and rotational inertias. The reader is referred to Brach [12], Goldsmith [31], Routh [64], Rubin [66], and Stewart [74] for discussions on other theories, applications and unresolved issues. For a discussion of some of Newton's contributions to this subject, see Problem 12 on pages 148–151 of [53]. There, Newton discusses the impact of two spheres.

6.3.1 The Model and Impact Stages

In what follows, we model two impacting bodies of masses m_1 and m_2 by two mass particles of masses m_1 and m_2, respectively. Furthermore, as the bodies are assumed to be in a state of purely translational motion, the velocity vector of any point of one of the bodies is identical to the velocity of the mass particle modeling the body. The position vector of the mass particle is defined to coincide with the center of mass of the body that it is modeling.

As summarized in Figure 6.4, we have four time periods to examine: I, just prior to impact; II, during the compression phase of the impact; III, during the restitution phase of the impact; and IV, immediately after the impact. This figure also summarizes the notation for the velocities that we use.

When the two bodies are in contact they are assumed to have a common unit normal $\mathbf{n}$ at the single point of contact. This vector is normal to the lateral surfaces of both bodies at the unique point of contact. We also define a set of orthonormal vectors $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$ by selecting $\mathbf{t}_1$ and $\mathbf{t}_2$ to be unit tangent vectors to the lateral surfaces of both bodies at the point of contact. The impact is assumed to be such that these vectors are constant for the duration of the collision.[4]

The impact is assumed to occur at time $t = t_0$. Stage II of the collision occurs during the time interval (t_0, t_1) and stage III occurs during the time interval $[t_1, t_2)$. At $t = t_2$, the bodies have just lost contact. At the end of stage II, the velocities of the bodies in the direction of $\mathbf{n}$ are assumed to be equal $(= v_{II})$.

6.3.2 Linear Impulses During Impact

Pertaining to the forces exerted by the bodies on each other during the impact, let $\mathbf{F}_{1_d}$ and $\mathbf{F}_{1_r}$ be the forces exerted by body 2 on body 1 during stages II and III, respectively. Similarly, we denote by $\mathbf{F}_{2_d}$ and $\mathbf{F}_{2_r}$ the forces exerted by body 1 on body 2 during stages II and III, respectively. All other forces acting on bodies 1 and 2 are assumed to have the resultants $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively.

The following three assumptions are made for the aforementioned forces. First, the impact is frictionless:

$$\mathbf{F}_{1_d} = F_{1_d}\mathbf{n}, \qquad \mathbf{F}_{1_r} = F_{1_r}\mathbf{n}, \qquad \mathbf{F}_{2_d} = F_{2_d}\mathbf{n}, \qquad \mathbf{F}_{2_r} = F_{2_r}\mathbf{n}.$$

That is, these forces have no tangential components. Second, the linear impulse of these forces dominates those due to $\mathbf{R}_1$ and $\mathbf{R}_2$:

$$\begin{aligned}
\int_{t_0}^{t_1} \left(\mathbf{F}_{1_d}(\tau) + \mathbf{R}_1(\tau)\right) d\tau &\approx \int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) d\tau, \\
\int_{t_1}^{t_2} \left(\mathbf{F}_{1_r}(\tau) + \mathbf{R}_1(\tau)\right) d\tau &\approx \int_{t_1}^{t_2} \mathbf{F}_{1_r}(\tau) d\tau, \\
\int_{t_0}^{t_1} \left(\mathbf{F}_{2_d}(\tau) + \mathbf{R}_2(\tau)\right) d\tau &\approx \int_{t_0}^{t_1} \mathbf{F}_{2_d}(\tau) d\tau, \\
\int_{t_1}^{t_2} \left(\mathbf{F}_{2_r}(\tau) + \mathbf{R}_2(\tau)\right) d\tau &\approx \int_{t_1}^{t_2} \mathbf{F}_{2_r}(\tau) d\tau.
\end{aligned}$$

[4] An example illustrating these three vectors is shown in Figure 6.8. We also note that for many problems these vectors will coincide with the Cartesian basis vectors.

It is normally assumed that the time interval $[t_0, t_2]$ is small in order for this assumption to hold. Finally, we assume equal and opposite collisional forces: $\mathbf{F}_{1_r} = -\mathbf{F}_{2_r}$ and $\mathbf{F}_{1_d} = -\mathbf{F}_{2_d}$.

We now define the coefficient of restitution e:

$$e = \frac{\int_{t_1}^{t_2} \mathbf{F}_{1_r}(\tau) \cdot \mathbf{n} d\tau}{\int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) \cdot \mathbf{n} d\tau} = \frac{\int_{t_1}^{t_2} \mathbf{F}_{2_r}(\tau) \cdot \mathbf{n} d\tau}{\int_{t_0}^{t_1} \mathbf{F}_{2_d}(\tau) \cdot \mathbf{n} d\tau}.$$

Here, we used the equal and opposite nature of the interaction forces. You should notice that if $e = 1$, then the linear impulse during the compression stage is equal to the linear impulse during the restitution phase. In this case, the collision is said to be perfectly elastic. If $e = 0$, then the linear impulse during the restitution phase is zero, and the collision is said to be perfectly plastic. In general, $0 \leq e \leq 1$, and e must be determined from an experiment.

To write the coefficient of restitution in a more convenient form using velocities, we first record the following integral forms of the balance of linear momentum for each particle in the direction of $\mathbf{n}$ during stages II and III:

$$\begin{aligned}
m_1 v_{II} - m_1 \mathbf{v}_1 \cdot \mathbf{n} &= \int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) \cdot \mathbf{n} d\tau, \\
m_2 v_{II} - m_2 \mathbf{v}_2 \cdot \mathbf{n} &= \int_{t_0}^{t_1} \mathbf{F}_{2_d}(\tau) \cdot \mathbf{n} d\tau, \\
m_1 \mathbf{v}_1^{'} \cdot \mathbf{n} - m_1 v_{II} &= \int_{t_1}^{t_2} \mathbf{F}_{1_r}(\tau) \cdot \mathbf{n} d\tau = e \int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) \cdot \mathbf{n} d\tau, \\
m_2 \mathbf{v}_2^{'} \cdot \mathbf{n} - m_2 v_{II} &= \int_{t_1}^{t_2} \mathbf{F}_{2_r}(\tau) \cdot \mathbf{n} d\tau = e \int_{t_0}^{t_1} \mathbf{F}_{2_d}(\tau) \cdot \mathbf{n} d\tau.
\end{aligned}$$

In these four equations v_{II} is the common velocity in the direction of $\mathbf{n}$ at the end of the compression phase, and the prime distinguishes the postimpact velocity vectors from their preimpact counterparts $\mathbf{v}_1$ and $\mathbf{v}_2$. From these four equations, we can determine the velocity v_{II}:

$$v_{II} = \frac{\mathbf{v}_1^{'} \cdot \mathbf{n} + e \mathbf{v}_1 \cdot \mathbf{n}}{1 + e} = \frac{\mathbf{v}_2^{'} \cdot \mathbf{n} + e \mathbf{v}_2 \cdot \mathbf{n}}{1 + e}.$$

We can also manipulate these four equations to find a familiar expression for the coefficient of restitution:

$$e = \frac{\mathbf{v}_2^{'} \cdot \mathbf{n} - \mathbf{v}_1^{'} \cdot \mathbf{n}}{\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n}}.$$

This equation is used by many authors as the definition of the coefficient of restitution.

6.3.3 Linear Momenta

We now consider the integral forms of the balance of linear momentum for each particle. The time interval of integration is the duration of the impact:

$$m_1\mathbf{v}_1^{'} - m_1\mathbf{v}_1 = \int_{t_0}^{t_1} \left(\mathbf{F}_{1_d}(\tau) + \mathbf{R}_1(\tau)\right) d\tau + \int_{t_1}^{t_2} \left(\mathbf{F}_{1_r}(\tau) + \mathbf{R}_1(\tau)\right) d\tau,$$
$$m_2\mathbf{v}_2^{'} - m_2\mathbf{v}_2 = \int_{t_0}^{t_1} \left(\mathbf{F}_{2_d}(\tau) + \mathbf{R}_2(\tau)\right) d\tau + \int_{t_1}^{t_2} \left(\mathbf{F}_{2_r}(\tau) + \mathbf{R}_2(\tau)\right) d\tau.$$

Using the coefficient of restitution e, the assumptions that $\mathbf{F}_{1_d} = -\mathbf{F}_{2_d}$, $\mathbf{F}_{1_r} = -\mathbf{F}_{2_r}$, and that the linear impulses of these forces dominate those due to $\mathbf{R}_1$ and $\mathbf{R}_2$, we find that

$$m_1\mathbf{v}_1^{'} - m_1\mathbf{v}_1 = (1+e)\int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) d\tau,$$
$$m_2\mathbf{v}_2^{'} - m_2\mathbf{v}_2 = -(1+e)\int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) d\tau.$$

We now take the three components of these equations with respect to the basis vectors $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$. The components of these equations in the tangential directions imply that the linear momenta of the particles in these directions are conserved:

$$\mathbf{v}_1^{'} \cdot \mathbf{t}_1 = \mathbf{v}_1 \cdot \mathbf{t}_1, \qquad \mathbf{v}_1^{'} \cdot \mathbf{t}_2 = \mathbf{v}_1 \cdot \mathbf{t}_2,$$
$$\mathbf{v}_2^{'} \cdot \mathbf{t}_1 = \mathbf{v}_2 \cdot \mathbf{t}_1, \qquad \mathbf{v}_2^{'} \cdot \mathbf{t}_2 = \mathbf{v}_2 \cdot \mathbf{t}_2.$$

In addition, from the two components in the $\mathbf{n}$ direction, we find that the linear momentum of the system in this direction is conserved:

$$m_2\mathbf{v}_2^{'} \cdot \mathbf{n} + m_1\mathbf{v}_1^{'} \cdot \mathbf{n} = m_2\mathbf{v}_2 \cdot \mathbf{n} + m_1\mathbf{v}_1 \cdot \mathbf{n},$$
$$m_2\mathbf{v}_2^{'} \cdot \mathbf{n} - m_2\mathbf{v}_2 \cdot \mathbf{n} = -(1+e)\int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) \cdot \mathbf{n} d\tau.$$

The previous six equations should be sufficient to determine the six postimpact velocities $\mathbf{v}_1^{'}$ and $\mathbf{v}_2^{'}$ provided that one knows the preimpact velocities and the linear impulse of $\mathbf{F}_{1_d}$ during the collision. However, this linear impulse is problem-dependent, and so one generally supplements these equations with the specification of the coefficient of restitution to determine the postimpact velocity vectors.

6.3.4 The Postimpact Velocities

It is convenient at this point to summarize the equations and show how they are used to solve certain problems. For the problems of interest one is often given e, $\mathbf{v}_1$, $\mathbf{v}_2$,

m_1, m_2, and $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$ and asked to calculate the postimpact velocity vectors $\mathbf{v}_1^{'}$ and $\mathbf{v}_2^{'}$.

From the previous two sets of equations and the specification of the coefficient of restitution, we see that we have one equation to determine the deformational linear impulse and six equations to determine the postimpact velocity vectors. With some algebra, one obtains

$$\mathbf{v}_1^{'} = (\mathbf{v}_1 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_1 \cdot \mathbf{t}_2)\mathbf{t}_2 + \frac{1}{m_1 + m_2}((m_1 - em_2)\mathbf{v}_1 \cdot \mathbf{n} + (1+e)m_2\mathbf{v}_2 \cdot \mathbf{n})\mathbf{n},$$
$$\mathbf{v}_2^{'} = (\mathbf{v}_2 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_2 \cdot \mathbf{t}_2)\mathbf{t}_2 + \frac{1}{m_1 + m_2}((m_2 - em_1)\mathbf{v}_2 \cdot \mathbf{n} + (1+e)m_1\mathbf{v}_1 \cdot \mathbf{n})\mathbf{n}.$$

You should notice how the postimpact velocity vectors depend on the mass ratios. We could also calculate the linear impulse of $\mathbf{F}_{1_d}$, but we do not pause to do so here.

It is important to remember that the above expressions for the postimpact velocities are consequences of the following: (i) the linear momenta of each particle in the tangential directions $\mathbf{t}_1$ and $\mathbf{t}_2$ are conserved during the impact, (ii) the combined linear momentum of the particles in the normal direction is conserved during the impact, and (iii) the coefficient of restitution e needs to be provided to determine the aforementioned velocity vectors.[5]

6.3.5 Kinetic Energy and the Coefficient of Restitution

Previously, we have used two equivalent definitions of the coefficient of restitution:

$$e = \frac{\mathbf{v}_2^{'} \cdot \mathbf{n} - \mathbf{v}_1^{'} \cdot \mathbf{n}}{\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n}},$$

and

$$e = \frac{\int_{t_1}^{t_2} \mathbf{F}_{1_r}(\tau) \cdot \mathbf{n} d\tau}{\int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) \cdot \mathbf{n} d\tau} = \frac{\int_{t_1}^{t_2} \mathbf{F}_{2_r}(\tau) \cdot \mathbf{n} d\tau}{\int_{t_0}^{t_1} \mathbf{F}_{2_d}(\tau) \cdot \mathbf{n} d\tau}.$$

Often, the change in kinetic energy is used to specify the coefficient of restitution. We now examine why this is possible.

[5] It should be clear that, given the preimpact velocity vectors, the balance laws give only six equations from which one needs to determine the six postimpact velocities and the linear impulses of $\mathbf{F}_{1_d}$ and $\mathbf{F}_{1_r}$. The introduction of the coefficient of restitution e and the assumption of a common normal velocity v_{II} at time $t = t_1$ gives two more equations which renders the system of equations solvable. That is, these two equations close the system of equations.

The kinetic energy of the system just prior to impact T and the kinetic energy immediately following the collision $T^{'}$ are, by definition,[6]

$$T = \frac{1}{2}m_1\mathbf{v}_1\cdot\mathbf{v}_1 + \frac{1}{2}m_2\mathbf{v}_2\cdot\mathbf{v}_2, \qquad T^{'} = \frac{1}{2}m_1\mathbf{v}_1^{'}\cdot\mathbf{v}_1^{'} + \frac{1}{2}m_2\mathbf{v}_2^{'}\cdot\mathbf{v}_2^{'}.$$

We recall that the collision changes only the $\mathbf{n}$ components of the velocity vectors. Hence,

$$T - T^{'} = \frac{1}{2}m_1\left((\mathbf{v}_1\cdot\mathbf{n})^2 - \left(\mathbf{v}_1^{'}\cdot\mathbf{n}\right)^2\right) + \frac{1}{2}m_2\left((\mathbf{v}_2\cdot\mathbf{n})^2 - \left(\mathbf{v}_2^{'}\cdot\mathbf{n}\right)^2\right).$$

Substituting for the normal components of the postimpact velocity vectors, we obtain the well-known equation

$$T - T^{'} = \frac{m_1 m_2}{2m_1 + 2m_2}\left(\mathbf{v}_1\cdot\mathbf{n} - \mathbf{v}_2\cdot\mathbf{n}\right)^2\left(1 - e^2\right).$$

Hence, if $-1 \leq e \leq 1$, the kinetic energy of the system cannot be increased as a result of the impact.

6.3.6 *Negative Values of the Coefficient of Restitution*

It follows from the previous equation that assuming e is negative does not preclude energy loss during an impact. Indeed, Brach [12] considers the example of a ball shattering and then passing through a window as an example of a problem where e is negative. In problems where e is negative,

$$e = \frac{\mathbf{v}_2^{'}\cdot\mathbf{n} - \mathbf{v}_1^{'}\cdot\mathbf{n}}{\mathbf{v}_1\cdot\mathbf{n} - \mathbf{v}_2\cdot\mathbf{n}} < 0,$$

and hence, $\mathbf{v}_1^{'}\cdot\mathbf{n} - \mathbf{v}_2^{'}\cdot\mathbf{n}$ and $\mathbf{v}_1\cdot\mathbf{n} - \mathbf{v}_2\cdot\mathbf{n}$ have the same sign. As a result, the colliding bodies must interpenetrate and pass through each other during the course of the impact. To eliminate this behavior, it is generally assumed that e is positive.

6.4 Impact of a Particle and a Massive Object

To illustrate the previous results, consider a particle of mass m_1 that collides with a massive object (see Figure 6.5). Intuitively, we expect that the velocity of the massive object will not be affected by the collision. The analysis below confirms this.

[6] The definition of the kinetic energy of a system of particles is discussed in further detail in Section 7.2.4 of Chapter 7.

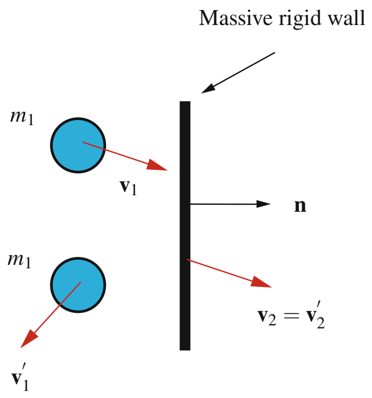

Fig. 6.5 *Impact of a particle and a massive object. The ratio of the masses* $m_1/(m_1+m_2) \approx 0$.

For the problem at hand, m_2 is assumed to be far greater than m_1:

$$\frac{m_1}{m_1+m_2} \approx 0, \qquad \frac{m_2}{m_1+m_2} \approx 1.$$

Substituting these results into the expressions for the postimpact velocities, we find that

$$\begin{aligned}\mathbf{v}_1' &= (\mathbf{v}_1 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_1 \cdot \mathbf{t}_2)\mathbf{t}_2 + (-e\mathbf{v}_1 \cdot \mathbf{n} + (1+e)\mathbf{v}_2 \cdot \mathbf{n})\mathbf{n}, \\ \mathbf{v}_2' &= (\mathbf{v}_2 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_2 \cdot \mathbf{t}_2)\mathbf{t}_2 + (\mathbf{v}_2 \cdot \mathbf{n})\mathbf{n} = \mathbf{v}_2.\end{aligned}$$

As expected, $\mathbf{v}_2$ is unaltered by the collision. You should also notice that if $e = 1$ and $\mathbf{v}_2 = \mathbf{0}$, then the particle rebounds with its velocity in the direction of $\mathbf{n}$ reversed, as expected. Finally, for a plastic collision, $e = 0$ and the velocity of m_1 in the direction of $\mathbf{n}$ attains the velocity of the massive object in this direction. We leave it as an exercise to calculate the energy lost in the collision. This exercise involves using the expression we previously established for $T - T'$.

6.5 A Bouncing Ball

As a further illustration of the general developments, consider the problem of a ball bouncing on a massive stationary surface (see Figure 6.6). We assume that the ball is launched with an initial velocity $\mathbf{v}_0$. The ball climbs to a height h_1 and then falls through the same distance and collides with the smooth surface. A rebound occurs and the ball then climbs to a height of h_2, and so on.

It what follows, we model the ball as a particle of mass m_1. Using the results from Section 6.4, we can consider the massive object as fixed. Furthermore, we assume that the coefficient of restitution e is a constant and less than 1. The model

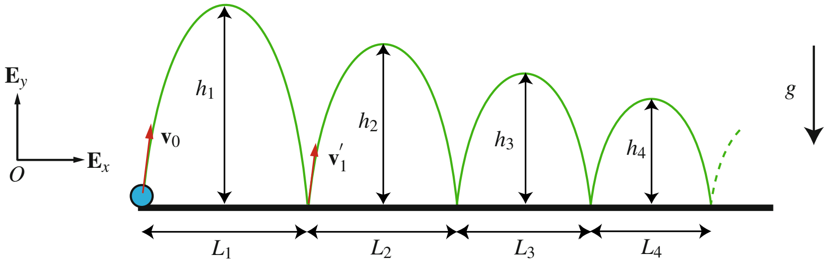

Fig. 6.6 *Schematic of a ball bouncing on a smooth horizontal surface.*

we develop predicts that although the ball will bounce an infinite number of times it will stop bouncing in a finite amount of time.[7]

As the particle is launched with an initial velocity of $\mathbf{v}_0 = v_{0_x}\mathbf{E}_x + v_{0_y}\mathbf{E}_y$, it is not too difficult to see that the ball rises to a maximum height h_1 and travels a distance L_1 before it impacts the surface. To compute h_1 and L_1, we ignore drag and use our earlier results from the projectile problem discussed in Section 1.6 in Chapter 1:

$$h_1 = \frac{v_{0_y}^2}{2g}, \qquad L_1 = \left(\frac{2v_{0_y}}{g}\right) v_{0_x}.$$

The velocity vector of the particle just prior to impact is $\mathbf{v}_1 = v_{0_x}\mathbf{E}_x - v_{0_y}\mathbf{E}_y$. With the help of the results of Section 6.4, we can conclude that the rebound velocity vector $\mathbf{v}_1'$ is

$$\mathbf{v}_1' = v_{0_x}\mathbf{E}_x + e v_{0_y}\mathbf{E}_y.$$

Furthermore, it is easy to infer that the particle rises to a maximum height h_2 and travels a distance L_2 before the next collision:

$$h_2 = \frac{e^2 v_{0_y}^2}{2g}, \qquad L_2 = \left(\frac{2e v_{0_y}}{g}\right) v_{0_x}.$$

Clearly,

$$h_2 = e^2 h_1, \qquad L_2 = eL_1.$$

Continuing with this analysis, we find that

$$h_i = e^{2(i-1)} h_1, \qquad L_i = e^{(i-1)} L_1 \qquad (i = 1, 2, 3, \ldots).$$

Notice that as $i \to \infty$, then $h_i \to 0$.

To determine how far the ball travels along the surface before it stops bouncing and starts moving at a constant speed v_{0_x}, we need to compute the infinite sum L of

[7] The model we discuss is a prototypical example in control theory for illustrating Zeno behavior (see, e.g., Liberzon [45]). The term Zeno behavior is in reference to the Greek philosopher Zeno of Elea and his famous dichotomy paradox. He is perhaps more famous for his paradox about the Tortoise and the Hare.

the distances L_i:

$$L = \sum_{i=1}^{\infty} L_i = \left(1 + e + e^2 + \cdots = \sum_{i=1}^{\infty} e^{(i-1)}\right) L_1 = \left(\frac{1}{1-e}\right) L_1$$
$$= \left(\frac{2v_{0_y}}{g(1-e)}\right) v_{0_x}.$$

It may come as a surprise to note that, although the number of impacts is infinite, the distance L is finite. Furthermore, the time $T = L/v_{0_x}$ taken for the ball to stop bouncing is also finite:

$$T = \frac{2v_{0_y}}{g(1-e)}.$$

This is the Zeno behavior we mentioned earlier.

One might ask the question of what would happen if the surface that the ball is bouncing on started to move vertically? Some answers to this question can be found in a 1982 paper by Holmes [37]. These answers are surprisingly complex. Indeed, depending on the motion of the surface and the coefficient of restitution, Holmes showed that the time between impacts could vary in a chaotic manner. His paper has generated considerable interest in, and inspired many published works on, the bouncing ball system during the past thirty years.

6.6 Collision of Two Spheres

Another example of interest is shown in Figure 6.7. There, a sphere of mass m_1 and radius R that is moving at a constant velocity $\mathbf{v}_1 = 100\mathbf{E}_x$ (meters per second) collides with a stationary sphere of radius r and mass $m_2 = 2m_1$.

At the instant of impact, the position vectors of the centers of mass of the spheres differ in height by an amount h. Given that the coefficient of restitution $e = 0.5$, one seeks to determine the velocity vectors of the spheres immediately following the impact.

It is first necessary to determine the normal and tangent vectors at the contact point of the spheres during the impact. Referring to Figure 6.8, we see that these

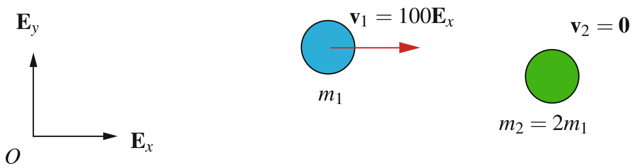

Fig. 6.7 *An impact of two spheres.*

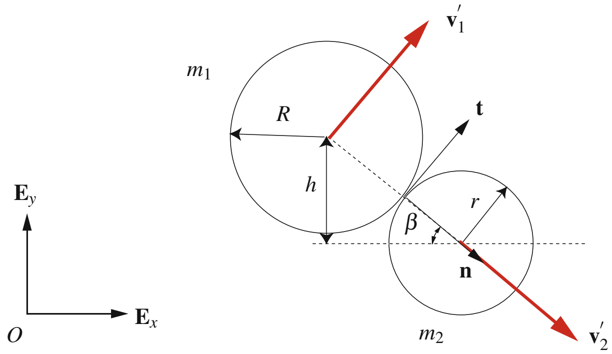

Fig. 6.8 *The geometry of the impacting spheres.*

vectors are

$$\mathbf{n} = \cos(\beta)\mathbf{E}_x - \sin(\beta)\mathbf{E}_y, \qquad \mathbf{t}_1 = \cos(\beta)\mathbf{E}_y + \sin(\beta)\mathbf{E}_x, \qquad \mathbf{t}_2 = \mathbf{E}_z.$$

Here, the angle β is defined by the relations

$$\sin(\beta) = \frac{h}{R+r}, \qquad \cos(\beta) = \frac{\sqrt{(r+R)^2 - h^2}}{r+R}.$$

Notice that this angle depends on h, r, and R.

Without calculating the postimpact velocity vectors, we can calculate the energy lost in the collision by using the formula for $T - T'$ which was established previously:

$$\begin{aligned} T - T' &= \frac{m_1 m_2}{2m_1 + 2m_2} (\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n})^2 \left(1 - e^2\right) \\ &= \frac{m_1}{3} (100 \cos(\beta))^2 \left(1 - e^2\right), \end{aligned}$$

where $e = 0.5$.

One method of solving this problem is to use the formulae given previously for the postimpact velocity vectors:

$$\begin{aligned} \mathbf{v}_1' &= (\mathbf{v}_1 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_1 \cdot \mathbf{t}_2)\mathbf{t}_2 + \frac{1}{m_1 + m_2} ((m_1 - em_2)\mathbf{v}_1 \cdot \mathbf{n} + (1+e)m_2 \mathbf{v}_2 \cdot \mathbf{n})\mathbf{n}, \\ \mathbf{v}_2' &= (\mathbf{v}_2 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_2 \cdot \mathbf{t}_2)\mathbf{t}_2 + \frac{1}{m_1 + m_2} ((m_2 - em_1)\mathbf{v}_2 \cdot \mathbf{n} + (1+e)m_1 \mathbf{v}_1 \cdot \mathbf{n})\mathbf{n}. \end{aligned}$$

Substituting for the values for the problem at hand,

$$\mathbf{v}_1 = 100\mathbf{E}_x, \qquad \mathbf{v}_2 = \mathbf{0}, \qquad e = 0.5, \qquad m_2 = 2m_1,$$

into these equations, we find that

$$\mathbf{v}_1^{'} = (\mathbf{v}_1 \cdot \mathbf{t}_1)\,\mathbf{t}_1 = 100\sin(\beta)\mathbf{t}_1,$$
$$\mathbf{v}_2^{'} = \frac{1}{2}(\mathbf{v}_1 \cdot \mathbf{n})\,\mathbf{n} = 50\cos(\beta)\mathbf{n}.$$

Notice that the initial velocity of the sphere of radius R in the direction of $\mathbf{n}$ has been annihilated by the collision (cf. Figure 6.8).

6.7 Summary

The first new concept in this chapter was the definition of the linear momentum $\mathbf{G} = m\mathbf{v}$. In addition, the integral form of the balance of linear momentum was introduced:

$$\mathbf{G}(t_1) - \mathbf{G}(t_0) = \int_{t_0}^{t_1} \mathbf{F}dt.$$

This equation is assumed to hold for all intervals of time and hence for all times t_0 and t_1. Because this balance law does not assume that $\mathbf{v}$ is differentiable, it is more general than $\mathbf{F} = m\mathbf{a}$.

The angular momentum $\mathbf{H}_O = \mathbf{r} \times \mathbf{G}$ of a particle relative to a fixed point O was introduced in Section 6.2. The time-rate of change of this momentum can be determined using the angular momentum theorem:

$$\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F}.$$

Associated with $\mathbf{G}$ and $\mathbf{H}_O$ are instances where certain components of these vectors are conserved. For a given vector $\mathbf{c}$, potential conservations of $\mathbf{G} \cdot \mathbf{c}$ and $\mathbf{H}_O \cdot \mathbf{c}$ can be established using the equations

$$\frac{d}{dt}(\mathbf{G} \cdot \mathbf{c}) = \mathbf{F} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = 0,$$
$$\frac{d}{dt}(\mathbf{H}_O \cdot \mathbf{c}) = (\mathbf{r} \times \mathbf{F}) \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = 0.$$

Several examples of linear and angular momenta conservations were discussed in the present chapter. Specifically, the conservation of a component of $\mathbf{G}$ for impact problems and projectile problems was covered. The more conceptually challenging conservation of a component of $\mathbf{H}_O$ was illustrated using a central force problem, Kepler's problem, and a particle moving on the surface of a smooth cone.

The majority of the chapter was devoted to a discussion of impact problems. Specifically, given two bodies of masses m_1 and m_2 whose respective preimpact velocity vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$, we wished to find the postimpact velocity vectors $\mathbf{v}_1^{'}$ and $\mathbf{v}_2^{'}$. The collisions of interest were restricted to cases where there was a unique point of contact with a well-defined unit normal vector $\mathbf{n}$. This normal vector was then used to construct a right-handed orthonormal triad $\{\mathbf{n}, \mathbf{t}_1, \mathbf{t}_2\}$. To solve the problems of interest, it was necessary to introduce a phenomenological constant, the coefficient of restitution e. In Sections 6.3.2 and 6.3.5, three alternative definitions of this constant were presented:

$$
\begin{aligned}
e &= \frac{\mathbf{v}_2^{'} \cdot \mathbf{n} - \mathbf{v}_1^{'} \cdot \mathbf{n}}{\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n}}, \\
e &= \frac{\int_{t_1}^{t_2} \mathbf{F}_{1_r}(\tau) \cdot \mathbf{n} d\tau}{\int_{t_0}^{t_1} \mathbf{F}_{1_d}(\tau) \cdot \mathbf{n} d\tau} = \frac{\int_{t_1}^{t_2} \mathbf{F}_{2_r}(\tau) \cdot \mathbf{n} d\tau}{\int_{t_0}^{t_1} \mathbf{F}_{2_d}(\tau) \cdot \mathbf{n} d\tau}, \\
T - T^{'} &= \frac{m_1 m_2}{2m_1 + 2m_2} (\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n})^2 \left(1 - e^2\right).
\end{aligned}
$$

We also noted that the restriction $0 \le e \le 1$ is normally imposed so as to preclude interpenetrability of the impacting bodies. If $e = 1$, the collision is said to be perfectly elastic and if $e = 0$, then the collision is said to be perfectly plastic.

Using the definition of the coefficient of restitution, conservation of linear momenta of each particle in the $\mathbf{t}_1$ and $\mathbf{t}_2$ directions, and conservation of the total linear momentum of the system of particles in the $\mathbf{n}$ direction, expressions were obtained for the postimpact velocity vectors:

$$
\begin{aligned}
\mathbf{v}_1^{'} &= (\mathbf{v}_1 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_1 \cdot \mathbf{t}_2)\mathbf{t}_2 + \frac{1}{m_1 + m_2}((m_1 - em_2)\mathbf{v}_1 \cdot \mathbf{n} + (1+e)m_2\mathbf{v}_2 \cdot \mathbf{n})\mathbf{n}, \\
\mathbf{v}_2^{'} &= (\mathbf{v}_2 \cdot \mathbf{t}_1)\mathbf{t}_1 + (\mathbf{v}_2 \cdot \mathbf{t}_2)\mathbf{t}_2 + \frac{1}{m_1 + m_2}((m_2 - em_1)\mathbf{v}_2 \cdot \mathbf{n} + (1+e)m_1\mathbf{v}_1 \cdot \mathbf{n})\mathbf{n}.
\end{aligned}
$$

The solutions of impact problems using these formulae were presented in Sections 6.4–6.6.

6.8 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

6.1. A particle of mass m is in motion on a smooth horizontal surface. Using $\mathbf{F} = m\mathbf{a}$, show that the resultant force $\mathbf{F}$ acting on the particle is zero and hence its linear momentum $\mathbf{G}$ remains constant.

6.2. For a particle of mass m that is falling under the influence of a gravitational force $-mg\mathbf{E}_z$, show that $\mathbf{G} \cdot \mathbf{E}_z$ is not conserved.

6.3. A particle of mass m is in motion on a smooth horizontal surface. Here, a gravitational force $-mg\mathbf{E}_z$ and an applied force $\mathbf{P} = P(t)\mathbf{E}_x$ acts on the particle. Using $\mathbf{F} = m\mathbf{a}$, show that $\mathbf{F} = \mathbf{P}$. Furthermore, show that

$$\mathbf{G}(t) = \mathbf{G}(t_0) + \left(\int_{t_0}^{t} P(\tau) d\tau \right) \mathbf{E}_x.$$

6.4. Starting from $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$, prove the angular momentum theorem: $\dot{\mathbf{H}}_O = \mathbf{r} \times \mathbf{F}$.

6.5. For a particle of mass m moving on a horizontal plane $z = 0$, show that $\mathbf{H}_O = mr^2\dot{\theta}\mathbf{E}_z$. When this momentum is conserved, show, with the aid of graphs of $\dot{\theta}$ as a function of r for various values of $h = \mathbf{H}_O \cdot \mathbf{E}_z$, that the sign of $\dot{\theta}$ cannot change. What does this imply about the motion of the particle?

6.6. Consider the example discussed in Section 6.2.5. Show that $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved when the spring is replaced by an inextensible string of length L. In your solution, assume that the string is being fed from an eyelet at O. Consequently, the length of string between O and the particle can change: $L = L(t)$.

6.7. Does the solution to Exercise 6.6 change if the angle $\alpha = 0$? To what physical problem does this case correspond?

6.8. Recall Kepler's problem discussed in Section 6.2.4. Show that angular momentum conservation can be used to reduce the equations governing θ and r to a single differential equation:

$$\ddot{r} - \frac{h^2}{m^2 r^3} + \frac{GMm}{r^2} = 0.$$

Show that this equation predicts a circular motion $r = r_0$, where

$$r_0 = \frac{h^2}{GMm^3}, \qquad \dot{\theta} = \frac{h}{mr_0^2}.$$

Finally, show that the velocity vector of the particle of mass m must be

$$\mathbf{v} = \frac{GMm^2}{h}\mathbf{e}_\theta.$$

Using the expression for r_0, estimate the angular momentum of the Earth orbiting the Sun.

6.9. With the help of the following result, which was established in Section 6.3.5,

$$T - T' = \frac{m_1 m_2}{2m_1 + 2m_2} (\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n})^2 \left(1 - e^2\right),$$

discuss the circumstances for which the kinetic energy loss is maximized or minimized in a collision.

6.10. Why is it necessary to know the coefficient of restitution e in order to solve an impact problem?

6.11. When two bodies are in contact at a point, there are two possible choices for $\mathbf{n}$: $\pm\mathbf{n}$. Why do the formulae for e, $\mathbf{v}_1^{'}$, and $\mathbf{v}_2^{'}$ that are presented in Section 6.7 give the same results for either choice of $\mathbf{n}$?

6.12. Recall the problem discussed in Section 6.4 where a particle of mass m impacts a massive object. If $e = 0$, verify that $\mathbf{v}_1^{'} \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n}$. On physical grounds why are the components of $\mathbf{v}_1$ in the $\mathbf{t}_1$ and $\mathbf{t}_2$ directions unaltered by the collision?

6.13. Using the results for the problem discussed in Section 6.4, show that if e is negative, then the particle passes through the massive object. When $e = -1$, show that the particle is unaffected by its collision with the massive object.

6.14. Using the results of the example discussed in Section 6.6, determine the postimpact velocity vectors of the spheres for the two cases $h = 0$ and $h = R + r$.

6.15. In order to examine aspects of the bouncing ball problem discussed in Section 6.5, I asked my daughter and stepson to drop a ball from a height of 3 feet, and we measured the successive rebound heights. The following three sets of estimates for h_i were recorded:

$$\begin{aligned}
&\text{Expt. 1:} \quad h_1 = 36, \quad h_2 = 30, \quad h_3 = 22,\\
&\text{Expt. 2:} \quad h_1 = 36, \quad h_2 = 27, \quad h_3 = 22.5,\\
&\text{Expt. 3:} \quad h_1 = 36, \quad h_2 = 25, \quad h_3 = 21.
\end{aligned}$$

The dimensions of h_i are in inches. Calculate an estimate of e for the impact of the ball with the ground.

6.16. For the bouncing ball problem discussed in Section 6.5, show that the total vertical distance $2H$ traveled by the particle before it stops bouncing is

$$2H = \left(\frac{v_{0_y}^2}{g}\right)\frac{1}{1-e^2}.$$

Give an interpretation of this result when $e = 0$.

Chapter 7
Dynamics of Systems of Particles

TOPICS

In this chapter, we continue the process of extending several results pertaining to a single particle to a system of particles. We start by defining the linear momentum, angular momenta, and kinetic energy for a system of particles. Next, we introduce a new concept, the center of mass C of a system of particles. A discussion of the conservation of kinematical quantities follows, which we illustrate with two detailed examples.

7.1 Preliminaries

We consider a system of n particles, each of mass m_i $(i = 1, \ldots, n)$. The position vector of the particle of mass m_i relative to a fixed point O is denoted by $\mathbf{r}_i$. Several quantities pertaining to the kinematics of this system are shown in Figure 7.1.

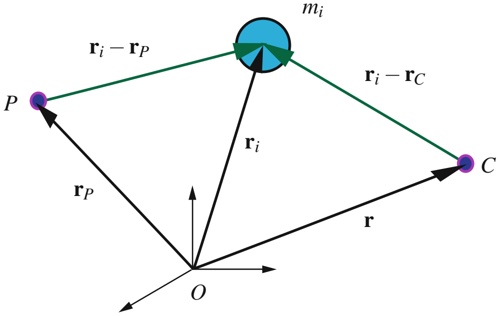

Fig. 7.1 *Some kinematical quantities pertaining to a particle of mass m_i.*

O.M. O'Reilly, *Engineering Dynamics: A Primer*, DOI 10.1007/978-1-4419-6360-4_7,

The velocity vector of the particle of mass m_i is defined as

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt}$$

and the (absolute) acceleration vector $\mathbf{a}_i$ of this particle is

$$\mathbf{a}_i = \frac{d^2\mathbf{r}_i}{dt^2} = \frac{d\mathbf{v}_i}{dt}.$$

We then have that the linear momentum $\mathbf{G}_i$ of the particle of mass m_i is

$$\mathbf{G}_i = m_i\mathbf{v}_i,$$

and the angular momentum $\mathbf{H}_{P_i}$ of the particle of mass m_i relative to a point P is

$$\mathbf{H}_{P_i} = (\mathbf{r}_i - \mathbf{r}_P) \times \mathbf{G}_i,$$

where $\mathbf{r}_P$ is the position vector of the point P relative to O. Finally, the kinetic energy T_i of the particle of mass m_i is

$$T_i = \frac{1}{2}m_i\mathbf{v}_i \cdot \mathbf{v}_i.$$

7.2 Center of Mass, Momenta, and Kinetic Energy

7.2.1 The Center of Mass

The center of mass C of the system of particles is defined as the point whose position vector $\mathbf{r}$ is defined by

$$\mathbf{r} = \frac{1}{m}\sum_{k=1}^{n} m_k\mathbf{r}_k,$$

where

$$m = \sum_{k=1}^{n} m_k$$

is the total mass of the system of particles.

The velocity $\mathbf{v}$ of the center of mass is obtained from the above equation by differentiating the expression for $\mathbf{r}$ with respect to time:

$$\mathbf{v} = \frac{1}{m}\sum_{k=1}^{n} m_k\mathbf{v}_k = \frac{1}{m}\sum_{k=1}^{n} \mathbf{G}_k.$$

You should notice that the velocity of the center of mass is a weighted sum of the velocities of the particles.

It is convenient to record the identities

$$\sum_{k=1}^{n} m_k(\mathbf{r}-\mathbf{r}_k)=\mathbf{0}, \qquad \sum_{k=1}^{n} m_k(\mathbf{v}-\mathbf{v}_k)=\mathbf{0}.$$

These identities are shortly used to derive convenient expressions for the linear and angular momenta and kinetic energy of a system of particles. The method of manipulation employed there is similar to that used later for rigid bodies.

7.2.2 Linear Momentum

The linear momentum $\mathbf{G}$ of the system of particles is the sum of the linear momenta of the individual particles. It follows from the definition of the center of mass that

$$\mathbf{G}=m\frac{d\mathbf{r}}{dt}=\sum_{k=1}^{n} m_k\frac{d\mathbf{r}_k}{dt}=\sum_{k=1}^{n}\mathbf{G}_k.$$

In words, the linear momentum of the center of mass is the linear momentum of the system.

7.2.3 Angular Momentum

Similarly, the angular momentum $\mathbf{H}_P$ of the system of particles relative to a point P, whose position vector relative to O is $\mathbf{r}_P$, is the sum of the individual angular momenta:

$$\mathbf{H}_P=\sum_{k=1}^{n}\mathbf{H}_{P_k}=\sum_{k=1}^{n}(\mathbf{r}_k-\mathbf{r}_P)\times m_k\mathbf{v}_k.$$

Using the definition of the center of mass, we can write $\mathbf{H}_P$ in a more convenient form:

$$\begin{aligned}\mathbf{H}_P &= \sum_{k=1}^{n}\mathbf{H}_{P_k}=\sum_{k=1}^{n}(\mathbf{r}_k-\mathbf{r}_P)\times\mathbf{G}_k\\ &= \sum_{k=1}^{n}(\mathbf{r}_k-\mathbf{r}+\mathbf{r}-\mathbf{r}_P)\times\mathbf{G}_k\\ &= \left(\sum_{k=1}^{n}(\mathbf{r}_k-\mathbf{r})\times\mathbf{G}_k\right)+(\mathbf{r}-\mathbf{r}_P)\times\left(\sum_{k=1}^{n}\mathbf{G}_k\right).\end{aligned}$$

That is,

$$\mathbf{H}_P=\mathbf{H}_C+(\mathbf{r}-\mathbf{r}_P)\times\mathbf{G}.$$

In this equation,

$$\mathbf{H}_C = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}) \times m_k \mathbf{v}_k$$

is the angular momentum of the system of particles relative to its center of mass C.

7.2.4 Kinetic Energy

The kinetic energy T of the system of particles is defined to be the sum of their individual kinetic energies:

$$T = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{2} m_k \mathbf{v}_k \cdot \mathbf{v}_k.$$

In general, the kinetic energy of the system is not equal to the kinetic energy of its center of mass. This shortly becomes evident.

Using the center of mass, T can be expressed in another form:

$$\begin{aligned} T = \sum_{k=1}^{n} T_k &= \frac{1}{2} \sum_{k=1}^{n} m_k \mathbf{v}_k \cdot \mathbf{v}_k \\ &= \frac{1}{2} \sum_{k=1}^{n} m_k \mathbf{v}_k \cdot \mathbf{v}_k - \mathbf{v} \cdot \sum_{k=1}^{n} m_k (\mathbf{v}_k - \mathbf{v}). \end{aligned}$$

Notice that we have added a term to the right-hand side that is equal to zero. Some minor manipulations of this result can then be used to show that

$$\begin{aligned} T &= \frac{1}{2} \sum_{k=1}^{n} m_k \mathbf{v}_k \cdot \mathbf{v}_k - \mathbf{v} \cdot \sum_{k=1}^{n} m_k (\mathbf{v}_k - \mathbf{v}) \\ &= \frac{1}{2} \sum_{k=1}^{n} m_k \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \sum_{k=1}^{n} m_k (\mathbf{v}_k \cdot \mathbf{v}_k - 2\mathbf{v}_k \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}). \end{aligned}$$

Completing the square, we obtain the final desired result:

$$T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \sum_{k=1}^{n} m_k (\mathbf{v}_k - \mathbf{v}) \cdot (\mathbf{v}_k - \mathbf{v}).$$

That is, the kinetic energy of a system of particles is the kinetic energy of its center of mass plus another term that is proportional to the magnitude of the velocity of the particles relative to the center of mass.[1]

[1] This result is sometimes known as the Koenig decomposition of the kinetic energy of a system of particles. In Section 9.2 of Chapter 9, the corresponding decomposition for a rigid body is discussed.

7.3 Kinetics of Systems of Particles

For each individual particle one has Euler's first law (which is equivalent to Newton's second law and the balance of linear momentum):

$$\mathbf{F}_i = m_i \mathbf{a}_i.$$

Adding these n equations and using the definition of the center of mass, we find that

$$\mathbf{F} = m\mathbf{a},$$

where $\mathbf{a} = \dot{\mathbf{v}}$ and $\mathbf{F}$ is the resultant force acting on the system of particles:

$$\mathbf{F} = \sum_{k=1}^{n} \mathbf{F}_k.$$

The equation $\mathbf{F} = m\mathbf{a}$ is very useful and allows us to solve for the motion of the center of mass of the system.

In many systems of particles problems, determining the motions of the particles, by solving the set of coupled second-order ordinary differential equations for $\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t)$, is an extremely difficult task[2] and one that is well beyond the scope of an undergraduate engineering dynamics course.

7.4 Conservation of Linear Momentum

We first consider conditions for the conservation of the component of the linear momentum $\mathbf{G}$ in the direction of a given vector $\mathbf{c} = \mathbf{c}(t)$. The result parallels the case for a single particle discussed in Section 6.1 of Chapter 6, and so we merely quote it. If $\mathbf{F} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = 0$, then $\mathbf{G} \cdot \mathbf{c}$ is conserved. In the next section, an example of this situation is discussed.

A second form of linear momentum conservation arises in the impact problems where the total linear momentum of the system is conserved. Examples of this type were explored in Sections 6.3–6.6 of Chapter 6.

7.5 The Cart and the Pendulum

As a first example of a system where conservation of linear momentum arises, consider the system shown in Figure 7.2. A particle of mass m_1 is attached by a spring of stiffness K and unstretched length L to another particle of mass m_2. A vertical gravitational force also acts on each particle. The mass m_1 is free to move on a smooth

[2] This can, perhaps, be appreciated by considering the examples discussed in Sections 7.5 and 7.7.

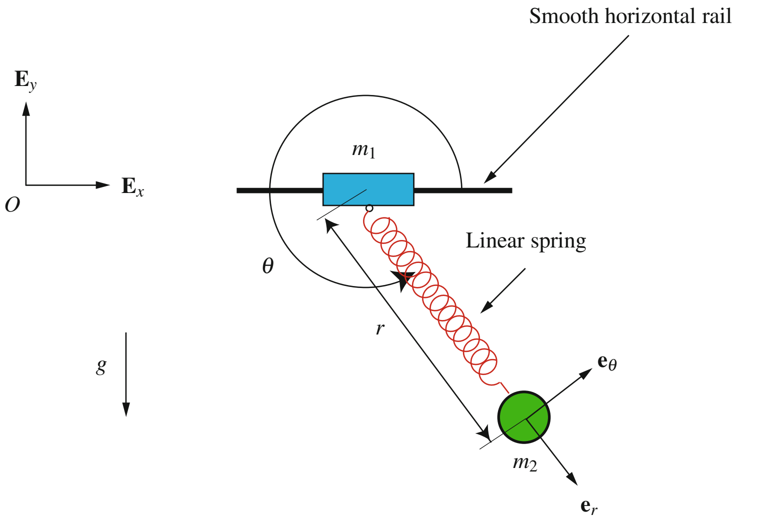

Fig. 7.2 *The cart and the pendulum.*

horizontal rail, and the second particle is free to move on a vertical plane. We wish to show that the linear momentum of the system in the $\mathbf{E}_x$ direction is conserved and then examine what this means for the motion of the system.

7.5.1 Kinematics

We first give some details on the kinematics of the system by defining the position vectors of both particles:

$$\mathbf{r}_1 = x\mathbf{E}_x + y_0\mathbf{E}_y + z_0\mathbf{E}_z, \qquad \mathbf{r}_2 = \mathbf{r}_1 + r\mathbf{e}_r,$$

where y_0 and z_0 are constants and the position vector of m_2 relative to m_1 is described using a cylindrical polar coordinate system. As expected, the position vector of the center of mass of the system lies at some point along the spring:

$$\bar{\mathbf{r}} = \frac{1}{m_1 + m_2}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) = \mathbf{r}_1 + \frac{m_2}{m_1 + m_2}r\mathbf{e}_r = \mathbf{r}_2 - \frac{m_1}{m_1 + m_2}r\mathbf{e}_r.$$

We can differentiate these position vectors in the usual manner to obtain the velocities and accelerations of the mass particles and the center of mass. Here, we record only the velocity vectors:

$$\mathbf{v}_1 = \dot{x}\mathbf{E}_x, \qquad \mathbf{v}_2 = \dot{x}\mathbf{E}_x + \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$

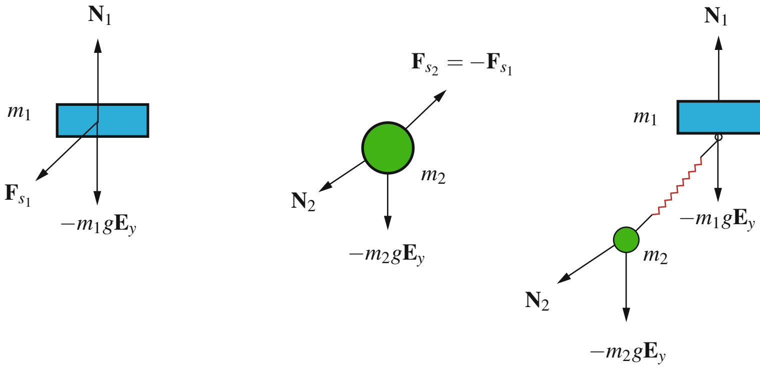

Fig. 7.3 *Free-body diagrams for the cart, pendulum, and system.*

$$\mathbf{v} = \dot{x}\mathbf{E}_x + \frac{m_2}{m_1 + m_2}\left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\right).$$

The acceleration vectors can be obtained in the usual manner. They are used below.

7.5.2 Forces

We now turn to the free-body diagrams for the individual particles and the system of particles. These are shown in Figure 7.3. The spring forces are (see Section 4.4 in Chapter 4)

$$\mathbf{F}_{s_1} = -\mathbf{F}_{s_2} = -K\left(\|\mathbf{r}_1 - \mathbf{r}_2\| - L\right)\frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} = K(r - L)\mathbf{e}_r.$$

You should note that these forces do not appear in the free-body diagram of the system. The normal forces acting on the particles are

$$\mathbf{N}_1 = N_{1_y}\mathbf{E}_y + N_{1_z}\mathbf{E}_z, \qquad \mathbf{N}_2 = N_{2_z}\mathbf{E}_z.$$

7.5.3 Balance Laws

We now consider the balances of linear momentum for the individual particles. For the particle of mass m_1 and the particle of mass m_2, we have, respectively,

$$-m_1 g\mathbf{E}_y + N_{1_y}\mathbf{E}_y + N_{1_z}\mathbf{E}_z + K(r - L)\mathbf{e}_r = m_1\ddot{x}\mathbf{E}_x,$$

$$-m_2 g\mathbf{E}_y + N_{2_z}\mathbf{E}_z - K(r - L)\mathbf{e}_r = m_2\ddot{x}\mathbf{E}_x + m_2(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + m_2(r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta.$$

These are six equations to determine the three unknown components of $\mathbf{N}_1$ and $\mathbf{N}_2$ and provide differential equations for r, x, and θ as functions of time.

With very little work, we find that

$$\mathbf{N}_1 = (m_1 g - K(r-L)\sin(\theta))\,\mathbf{E}_y, \qquad \mathbf{N}_2 = \mathbf{0}.$$

Next, we obtain a set of nonlinear second-order ordinary differential equations:

$$\begin{aligned} m_1\ddot{x} &= K(r-L)\cos(\theta), \\ m_2\ddot{x}\cos(\theta) + m_2(\ddot{r} - r\dot{\theta}^2) &= -K(r-L) - m_2 g\sin(\theta), \\ -m_2\ddot{x}\sin(\theta) + m_2(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= -m_2 g\cos(\theta). \end{aligned}$$

Notice how the motions of the two particles are coupled. Given a set of initial conditions, $r(t_0)$, $\theta(t_0)$, $x(t_0)$, $\dot{r}(t_0)$, $\dot{\theta}(t_0)$, and $\dot{x}(t_0)$, the solution $r(t)$, $\theta(t)$, and $x(t)$ of these equations can be used to determine the motions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ of the particles. Such an analysis is beyond the scope of an undergraduate engineering dynamics course.

7.5.4 Analysis

Next, we consider the balance of linear momentum for the system of particles:[3]

$$-(m_2 + m_1)g\mathbf{E}_y + \mathbf{N}_1 + \mathbf{N}_2 = (m_1 + m_2)\ddot{x}\mathbf{E}_x + m_2(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + m_2(r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta.$$

We see immediately that $\mathbf{F}\cdot\mathbf{E}_x = \mathbf{0}$.[4] In other words, the linear momentum of the system in the $\mathbf{E}_x$ direction is conserved. This momentum is

$$\begin{aligned} \mathbf{G}\cdot\mathbf{E}_x &= (m_1 + m_2)\mathbf{v}\cdot\mathbf{E}_x \\ &= (m_1 + m_2)\dot{x} + m_2(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)). \end{aligned}$$

As a result of this conservation, the velocity of the center of mass in the $\mathbf{E}_x$ direction is constant, and the masses m_1 and m_2 move in such a manner as to preserve this constant velocity. If the initial value of $\mathbf{G}\cdot\mathbf{E}_x$ is equal to G_0, then the motion of the particle of mass m_1 is such that

$$\dot{x} = \frac{1}{m_1 + m_2}\left(G_0 - m_2(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta))\right).$$

[3] Due to the equal and opposite nature of the spring forces acting on the particles, this equation is equivalent to the addition of the balances of linear momentum for the individual particles. Because the spring is assumed to be massless, the equal and opposite nature of the spring forces can also be interpreted using Newton's third law.

[4] That is, $\mathbf{c} = \mathbf{E}_x$.

We return to this example later on to show that the total energy of this system is conserved.[5]

7.6 Conservation of Angular Momentum

In Section 7.2 we noted that the angular momentum of the system of particles relative to an arbitrary point P is

$$\mathbf{H}_P = \mathbf{H}_C + (\mathbf{r} - \mathbf{r}_P) \times \mathbf{G},$$

where

$$\mathbf{H}_C = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}) \times m_k \mathbf{v}_k$$

is the angular momentum of the system of particles relative to its center of mass C.

We first calculate an expression for its rate of change with respect to time:

$$\dot{\mathbf{H}}_P = \sum_{k=1}^{n} ((\mathbf{v}_k - \mathbf{v}) \times m_k \mathbf{v}_k + (\mathbf{r}_k - \mathbf{r}) \times m_k \mathbf{a}_k) + (\mathbf{v} - \mathbf{v}_P) \times \mathbf{G} + (\mathbf{r} - \mathbf{r}_P) \times \dot{\mathbf{G}},$$

where we have used the product rule. Invoking the balances of linear momentum, $\mathbf{F}_i = m_i \mathbf{a}_i$ and $\mathbf{F} = m\mathbf{a} = \dot{\mathbf{G}}$, the above equation becomes

$$\dot{\mathbf{H}}_P = \sum_{k=1}^{n} ((\mathbf{v}_k - \mathbf{v}) \times m_k \mathbf{v}_k + (\mathbf{r}_k - \mathbf{r}) \times \mathbf{F}_k) + (\mathbf{v} - \mathbf{v}_P) \times \mathbf{G} + (\mathbf{r} - \mathbf{r}_P) \times \dot{\mathbf{G}}.$$

We next eliminate those terms that are zero on the right-hand side of this equation with the partial assistance of the identities

$$\mathbf{v} \times \mathbf{G} = \mathbf{0}, \qquad \sum_{k=1}^{n} \mathbf{v} \times m_k \mathbf{v}_k = \mathbf{v} \times m\mathbf{v} = \mathbf{0}.$$

The final result now follows:

$$\dot{\mathbf{H}}_P = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}_P) \times \mathbf{F}_k - \mathbf{v}_P \times \mathbf{G}.$$

This result is known as the angular momentum theorem for a system of particles. We note that the resultant moment of the system of forces relative to P is

$$\mathbf{M}_P = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}_P) \times \mathbf{F}_k.$$

[5] It is a good exercise to convince yourself that a similar result for linear momentum conservation applies when one replaces the spring with a rigid rod of length L.

With this notation, the angular momentum theorem has a more compact form:

$$\dot{\mathbf{H}}_P = \mathbf{M}_P - \mathbf{v}_P \times \mathbf{G}.$$

It is very important to notice that $\dot{\mathbf{H}}_P$ is not necessarily equal to $\mathbf{M}_P$.

There are two important special cases of the angular momentum theorem. First, when P is a fixed point O, where $\mathbf{r}_O = \mathbf{0}$, then

$$\dot{\mathbf{H}}_O = \mathbf{M}_O, \text{ where } \mathbf{M}_O = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{F}_k$$

and $\mathbf{M}_O$ is the resultant moment relative to O. The second case arises when P is the center of mass C. In this case, as $\mathbf{v} \times \mathbf{G} = \mathbf{0}$, the expression for the rate of change of angular momentum simplifies to

$$\dot{\mathbf{H}}_C = \mathbf{M}_C, \text{ where } \mathbf{M}_C = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}) \times \mathbf{F}_k$$

and $\mathbf{M}_C$ is the resultant moment relative to C. For both of these cases we have the usual interpretation that the rate of change of angular momentum relative to a point is the resultant moment due to the forces acting on each particle. However, for the arbitrary case where P may be moving this interpretation does not hold.

We next consider the conditions whereby a component of the angular momentum $\mathbf{H}_P$ in the direction of a given vector $\mathbf{c} = \mathbf{c}(t)$ is conserved. The result parallels that for a single particle. However, here we allow for the possibility that P is moving. For a given vector $\mathbf{c}$, which may be a function of time, we wish to determine when

$$\frac{d}{dt}(\mathbf{H}_P \cdot \mathbf{c}) = 0.$$

Using the previous results, we find that for this conservation it is necessary and sufficient that

$$(\mathbf{M}_P - \mathbf{v}_P \times \mathbf{G}) \cdot \mathbf{c} + \mathbf{H}_P \cdot \dot{\mathbf{c}} = 0.$$

For a given problem and a specific point P, it is very difficult to find $\mathbf{c}$ such that this equation holds. In most posed problems at the undergraduate level, P is either the center of mass C or an origin O and $\mathbf{c} = \mathbf{E}_z$.

7.7 A System of Four Particles

The main class of problems where angular momentum conservation is useful is the mechanism shown in Figure 7.4. Here, four particles are attached to a vertical axle, by springs of stiffness K_i and unstretched length L_i ($i = 1, 2, 3,$ or 4). The particles are free to move on smooth horizontal rails. The rails and axle are free to rotate about

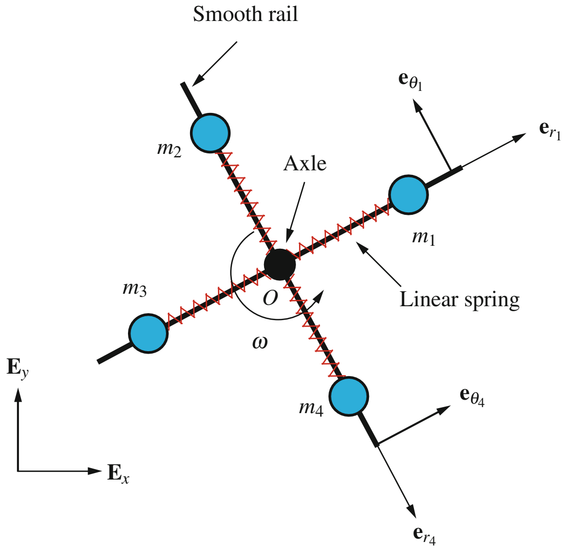

Fig. 7.4 *A system of four particles.*

the vertical with an angular speed ω. We now examine why $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved in this problem.

7.7.1 Kinematics

For each of the four particles, one defines a cylindrical polar coordinate system $\{r_i, \theta_i, z_i\}$. In particular, $d\theta_i/dt = \omega$, $\theta_2 - \theta_1 = \pi/2$, $\theta_3 - \theta_2 = \pi/2$, $\theta_4 - \theta_3 = \pi/2$, and $\theta_1 - \theta_4 = \pi/2$. We then have some familiar results:

$$\mathbf{r}_i = r_i \mathbf{e}_{r_i} + z_0 \mathbf{E}_z, \qquad \mathbf{v}_i = \dot{r}_i \mathbf{e}_{r_i} + r_i \omega \mathbf{e}_{\theta_i},$$

where $z_0 = 0$. The angular momentum of the system relative to the fixed point O is easily calculated using the definition of this angular momentum to be

$$\mathbf{H}_O = \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2\right) \omega \mathbf{E}_z.$$

7.7.2 Forces and Balance Laws

Leaving the free-body diagrams as an exercise, the resultant force on each particle is

$$\mathbf{F}_i = -K_i \left(r_i - L_i\right) \mathbf{e}_{r_i} + (N_{\theta_i}) \mathbf{e}_{\theta_i} + (N_{i_z} - m_i g) \mathbf{E}_z,$$

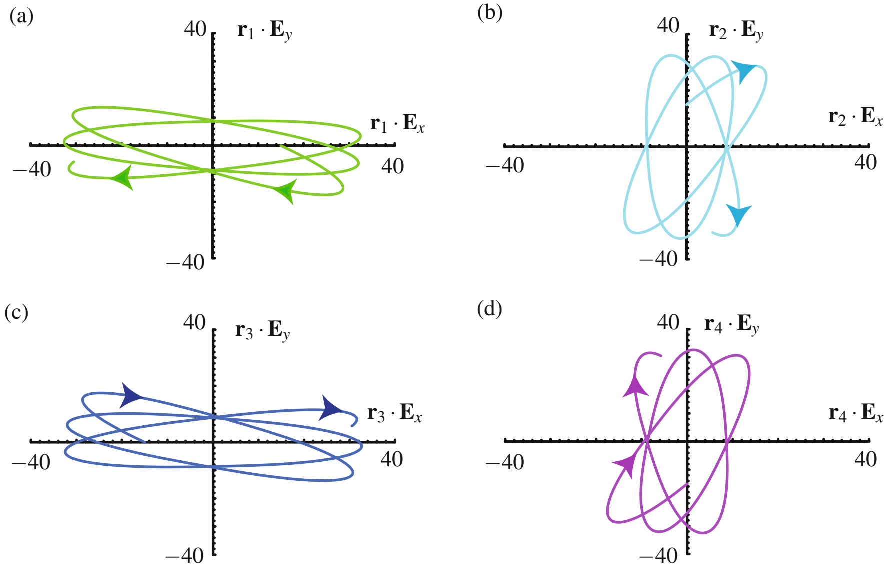

Fig. 7.5 *An example of the motions of the four particles for the system shown in Figure 7.4: (a) motion of m_1, (b) motion of m_2, (c) motion of m_3, and (d) motion of m_4.*

where $i = 1, 2, 3$, or 4.

From $\mathbf{F}_i = m_i\mathbf{a}_i$, one finds that $N_{i_z} = m_i g$, as expected. Performing a balance of angular momentum relative to O of the rails and axle in the $\mathbf{E}_z$ direction, we find, on ignoring the inertias of the rails and axle, that if there is no applied moment in the $\mathbf{E}_z$ direction, then[6]

$$r_1 N_{\theta_1} + r_2 N_{\theta_2} + r_3 N_{\theta_3} + r_4 N_{\theta_4} = 0.$$

7.7.3 Analysis

We are now in a position to show that $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{H}_O \cdot \mathbf{E}_z) &= \sum_{k=1}^{4} (\mathbf{r}_k \times \mathbf{F}_k) \cdot \mathbf{E}_z \\
&= \sum_{k=1}^{4} (\mathbf{r}_k \times (-K_k(r_k - L_k)\mathbf{e}_{r_k} + N_{\theta_k}\mathbf{e}_{\theta_k})) \cdot \mathbf{E}_z \\
&= r_1 N_{\theta_1} + r_2 N_{\theta_2} + r_3 N_{\theta_3} + r_4 N_{\theta_4} = 0.
\end{aligned}$$

[6] Strictly speaking, the axle and rails constitute a rigid body, so this result becomes clearer when we deal with kinematics of rigid bodies later on.

It follows that

$$\mathbf{H}_O \cdot \mathbf{E}_z = \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2\right) \omega$$

is constant. Here, if one knows the locations of the particles and the speed ω at one instant, then, for example, given the locations of the particles at a later time one can determine ω.

An example of the motion of the system of four particles where $\mathbf{H}_O$ is conserved is shown in Figure 7.5. This figure was obtained by numerically integrating the $\mathbf{e}_{r_i}$ components of the balances of linear momenta for each of the four particles, and using the conservation of $\mathbf{H}_O$ to determine ω. Integrating ω with respect to time then provides $\theta_{1,2,3,4}(t)$.

7.8 Work, Energy, and Conservative Forces

We first recall that for each particle in a system of n particles, we have the work-energy theorem: $\dot{T}_k = \mathbf{F}_k \cdot \mathbf{v}_k$. Recalling that the kinetic energy T of the system of particles is the sum of the individual kinetic energies, one immediately has the work-energy theorem for the system of particles:

$$\frac{dT}{dt} = \sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{v}_k.$$

We can start here and, paralleling the developments of Chapter 5, develop a theorem for the conservation of the total energy of the system of particles. To do this, it is convenient to decompose each $\mathbf{F}_k$ into the sum of conservative and nonconservative parts:

$$\mathbf{F}_k = \mathbf{F}_{nc_k} + \mathbf{F}_{c_k}, \qquad \mathbf{F}_{c_k} = -\frac{\partial U}{\partial \mathbf{r}_k},$$

where $U = U(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ is the total potential energy of the system. Thus,

$$\begin{aligned} \frac{dT}{dt} &= \sum_{k=1}^{n} \left(\mathbf{F}_{c_k} + \mathbf{F}_{nc_k}\right) \cdot \mathbf{v}_k \\ &= -\underbrace{\left(\sum_{k=1}^{n} \frac{\partial U}{\partial \mathbf{r}_k} \cdot \mathbf{v}_k\right)}_{=\dot{U}} + \sum_{k=1}^{n} \mathbf{F}_{nc_k} \cdot \mathbf{v}_k. \end{aligned}$$

Taking $\dot{U}$ to the left-hand side and defining the total energy $E = T + U$, we conclude that

$$\dot{E} = \sum_{k=1}^{n} \mathbf{F}_{nc_k} \cdot \mathbf{v}_k.$$

This alternative form of the work-energy theorem is very useful in applications.

We next examine two systems that were previously discussed and show how the total energy is conserved. We then make some general comments at the conclusion of this section.

7.8.1 The Cart and the Pendulum

We first consider the system of two particles discussed in Section 7.5. For this system, the work-energy theorem gives

$$\frac{dT}{dt} = (\mathbf{F}_{s_1} - m_1 g \mathbf{E}_y + \mathbf{N}_1) \cdot \mathbf{v}_1 + (\mathbf{F}_{s_2} - m_2 g \mathbf{E}_y + \mathbf{N}_2) \cdot \mathbf{v}_2.$$

Now, the normal forces are perpendicular to the velocities: $\mathbf{N}_1 \cdot \mathbf{v}_1 = \mathbf{N}_2 \cdot \mathbf{v}_2 = 0$. Furthermore,[7]

$$\begin{aligned} \mathbf{F}_{s_1} \cdot \mathbf{v}_1 + \mathbf{F}_{s_2} \cdot \mathbf{v}_2 &= -K(\|\mathbf{r}_1 - \mathbf{r}_2\| - L) \left(\frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \right) \cdot (\mathbf{v}_1 - \mathbf{v}_2) \\ &= -\frac{d}{dt} \left(\frac{K}{2} (\|\mathbf{r}_1 - \mathbf{r}_2\| - L)^2 \right). \end{aligned}$$

In summary,

$$\frac{dT}{dt} = -\frac{d}{dt} \left(\frac{K}{2} (\|\mathbf{r}_1 - \mathbf{r}_2\| - L)^2 + m_1 g \mathbf{E}_y \cdot \mathbf{r}_1 + m_2 g \mathbf{E}_y \cdot \mathbf{r}_2 \right).$$

It follows that the total energy of the system of particles is conserved:

$$\frac{d}{dt} \left(E = T + \frac{K}{2} (\|\mathbf{r}_1 - \mathbf{r}_2\| - L)^2 + m_1 g \mathbf{E}_y \cdot \mathbf{r}_1 + m_2 g \mathbf{E}_y \cdot \mathbf{r}_2 \right) = 0.$$

We can modify the cart-pendulum system by replacing the spring by a rigid massless rod of length L. The total energy of the modified system will again be conserved. We now show this. For the modified system, one has the kinematical results

$$\mathbf{r}_2 - \mathbf{r}_1 = L\mathbf{e}_r, \qquad \mathbf{v}_2 - \mathbf{v}_1 = L\dot{\theta}\mathbf{e}_\theta, \qquad (\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) = 0.$$

[7] To establish this result, one uses results from Section 5.5 of Chapter 5, replacing $\mathbf{r}_D$ with $\mathbf{r}_2$ and noting that $\mathbf{v}_2 \neq \mathbf{0}$. In addition, the identity

$$\frac{d\,\|\mathbf{x}\|}{dt} = \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{\|\mathbf{x}\|}$$

discussed in Section 5.5.2 is used.

Starting from the work-energy theorem

$$\frac{dT}{dt} = (S\mathbf{e}_r - m_1 g\mathbf{E}_y + \mathbf{N}_1)\cdot\mathbf{v}_1 + (-S\mathbf{e}_r - m_2 g\mathbf{E}_y + \mathbf{N}_2)\cdot\mathbf{v}_2.$$

Here, $S\mathbf{e}_r$ is the tension force in the rod. Again, the normal forces are perpendicular to the velocity vectors and, with the help of the kinematical results above, we can easily conclude energy conservation:

$$\begin{aligned}\frac{dE}{dt} = \frac{d}{dt}\left(T + m_1 g\mathbf{E}_y\cdot\mathbf{r}_1 + m_2 g\mathbf{E}_y\cdot\mathbf{r}_2\right) &= S\mathbf{e}_r\cdot(\mathbf{v}_1 - \mathbf{v}_2)\\ &= S\mathbf{e}_r\cdot\left(L\dot{\theta}\mathbf{e}_\theta\right) = 0.\end{aligned}$$

Notice that the tension force does work on each of the particles. However, its combined power is zero.

7.8.2 A System of Four Particles

Our final example is the system discussed in Section 7.7. Here, the work-energy theorem is

$$\begin{aligned}\frac{dT}{dt} &= \sum_{i=1}^{4}\mathbf{F}_i\cdot\mathbf{v}_i\\ &= \sum_{i=1}^{4}\left(-K_i\left(r_i - L_i\right)\mathbf{e}_{r_i} + (N_{\theta_i})\mathbf{e}_{\theta_i} + \left(N_{i_z} - m_i g\right)\mathbf{E}_z\right)\cdot\mathbf{v}_i.\end{aligned}$$

Simplifying the right-hand side of this equation, we obtain the result

$$\frac{dT}{dt} = \sum_{i=1}^{4}\left(-K_i(r_i - L_i)\dot{r}_i + \left(r_i N_{\theta_i}\right)\omega\right).$$

We note that

$$K_i(r_i - L_i)\dot{r}_i = \frac{d}{dt}\left(\frac{K_i}{2}(r_i - L_i)^2\right).$$

Furthermore, after recalling the result that

$$r_1 N_{\theta_1} + r_2 N_{\theta_2} + r_3 N_{\theta_3} + r_4 N_{\theta_4} = 0,$$

we conclude that

$$\frac{d}{dt}\left(E = \frac{1}{2}\sum_{i=1}^{4} m_i\mathbf{v}_i\cdot\mathbf{v}_i + K_i\left(\|\mathbf{r}_i\| - L_i\right)^2\right) = 0.$$

In other words, the total energy of the system is conserved.

7.8.3 Comment

In problems involving systems of particles, one uses energy conservation in an identical manner as in dealing with a single particle. A subtle feature of systems of particles is that the energy of the individual particles may not be conserved, but the sum of their energies is. This feature is present in the examples discussed above. It is a good exercise to repeat the analyses of energy conservation for the examples we have just considered using the alternative form of the work-energy theorem: $\dot{E} = \sum_{k=1}^{n} \mathbf{F}_{nc_k} \cdot \mathbf{v}_k$.

7.9 Summary

This chapter was devoted to the kinematics and kinetics of a system of n particles. The first new concept that was introduced was the center of mass C of the system of particles:

$$\mathbf{r} = \frac{1}{m} \sum_{k=1}^{n} m_k \mathbf{r}_k,$$

where $m = \sum_{k=1}^{n} m_k$ is the total mass of the system of particles. We then described how the linear momentum $\mathbf{G}$ of the system of particles was equal to the sum of their linear momenta:

$$\mathbf{G} = m\mathbf{v} = m\dot{\mathbf{r}} = \sum_{k=1}^{n} m_k \mathbf{v}_k.$$

Following this, the angular momentum $\mathbf{H}_P$ of the system of particles was shown to be the angular momentum of the center of mass plus the angular momentum $\mathbf{H}_C$ of the system of particles relative to their center of mass:

$$\mathbf{H}_P = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}_P) \times m_k \mathbf{v}_k = (\mathbf{r} - \mathbf{r}_P) \times m\mathbf{v} + \mathbf{H}_C,$$

where

$$\mathbf{H}_C = \sum_{k=1}^{n} (\mathbf{r}_k - \mathbf{r}) \times m_k \mathbf{v}_k.$$

Finally, the kinetic energy T of the system of particles was defined to be the sum of the kinetic energies of the individual particles:

$$T = \frac{1}{2} \sum_{k=1}^{n} m_k \mathbf{v}_k \cdot \mathbf{v}_k = \frac{1}{2} m\mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \sum_{k=1}^{n} m_k (\mathbf{v}_k - \mathbf{v}) \cdot (\mathbf{v}_k - \mathbf{v}).$$

In the expressions for T given above, we have also recorded the expression for the kinetic energy in terms of the kinetic energy of the center of mass and the "relative" kinetic energy of the particles relative to the center of mass.

Once the kinematical quantities for a system of particles were defined, their kinetics was discussed. By combining the balances of linear momentum for each particle, it was shown that

$$\mathbf{F} = \sum_{k=1}^{n} \mathbf{F}_k = \dot{\mathbf{G}} = m\dot{\mathbf{v}}.$$

This equation was used in Section 7.4 to establish a linear momentum conservation result. The following angular momentum theorems were established in Section 7.6:

$$\dot{\mathbf{H}}_O = \mathbf{M}_O, \qquad \dot{\mathbf{H}}_C = \mathbf{M}_C,$$

where O is a fixed point. These results were then used to show when a component of an angular momentum was conserved.

In Section 7.8, the work-energy theorem for a system of particles was discussed:

$$\dot{T} = \sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{v}_k.$$

This theorem was the starting point for proving energy conservation for systems of particles. From this theorem, it was shown that

$$\dot{E} = \sum_{k=1}^{n} \mathbf{F}_{nc_k} \cdot \mathbf{v}_k,$$

where $\mathbf{F}_{nc_k}$ is the nonconservative force acting on the kth particle and E is the total energy of the system of particles.

7.10 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

7.1. Starting from the definition of the position vector of the center of mass, show that

$$\sum_{k=1}^{n} m_k (\mathbf{r}_k - \mathbf{r}) = \mathbf{0}, \qquad \sum_{k=1}^{n} m_k (\mathbf{v}_k - \mathbf{v}) = \mathbf{0}.$$

Where were these identities used?

7.2. Starting from the definition of the angular momentum of a system of particles relative to a point P, prove that

$$\mathbf{H}_P = (\mathbf{r} - \mathbf{r}_P) \times m\mathbf{v} + \mathbf{H}_C.$$

7.3. Starting from the definition of the kinetic energy T of a system of particles, show that

$$T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \sum_{k=1}^{n} m_k \left(\mathbf{v}_k - \mathbf{v} \right) \cdot \left(\mathbf{v}_k - \mathbf{v} \right).$$

Using this result, show that the kinetic energy of a system of particles is not, in general, equal to the kinetic energy of the center of mass.

7.4. Consider two particles that are free to move on a horizontal surface $z = 0$. Vertical gravitational forces $-m_1 g \mathbf{E}_z$ and $-m_2 g \mathbf{E}_z$ act on the respective particles. The position vectors of the particles are

$$\mathbf{r}_1 = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{r}_2 = \mathbf{r}_1 + r\mathbf{e}_r.$$

Derive an expression for the position vector $\mathbf{r}$ of the center of mass C of this system of particles. Verify your answer by examining the limiting cases that m_1 is much larger than m_2 and vice versa.

7.5. Consider the system of particles discussed in Exercise 7.4. Suppose the particles are connected by a linear spring of stiffness K and unstretched length L. Show that the linear momenta $\mathbf{G} \cdot \mathbf{E}_x$ and $\mathbf{G} \cdot \mathbf{E}_y$ are conserved. What do these results imply about the motion of the center of mass C of this system of particles?

7.6. For the system of particles discussed in Exercise 7.5, prove that $\mathbf{H}_C \cdot \mathbf{E}_z$ is conserved. What does this result imply about $\dot{\theta}$?

7.7. Consider the system of particles discussed in Exercise 7.5. Starting from the work-energy theorem, prove that the total energy E of the system of particles is conserved. Here,

$$E = \frac{1}{2} \left(m_1 \mathbf{v}_1 \cdot \mathbf{v}_1 + m_2 \mathbf{v}_2 \cdot \mathbf{v}_2 \right) + \frac{K}{2} \left(r - L \right)^2.$$

7.8. For the cart and pendulum system discussed in Section 7.5, show that $\mathbf{G} \cdot \mathbf{E}_x$ and the total energy E are still conserved if the spring is replaced by an inextensible string of length L.

7.9. Consider the system of four particles discussed in Section 7.7. If one had the ability to measure r_1, r_2, r_3, r_4, and ω for this system, how would one verify that $\mathbf{H}_O \cdot \mathbf{E}_z$ was conserved?

7.10. Referring to the system of four particles discussed in Section 7.7, what are the $\mathbf{e}_{r_i}$ and $\mathbf{e}_{\theta_i}$ components of the balances of linear momenta for each of the four particles? How could the resulting differential equations and conservation of $\mathbf{H}_O$ be used to compute the motions of the four particles (cf. Figure 7.5)?

Part III
Dynamics of a Single Rigid Body

Chapter 8
Planar Kinematics of Rigid Bodies

TOPICS

In this chapter background material on the planar kinematics of rigid bodies is presented. In particular, we show how to establish certain useful representations for the velocity and acceleration vectors of any material point of a rigid body. We also discuss the angular velocity vector of a rigid body. These concepts are illustrated using two important applications: mechanisms and rolling rigid bodies. Finally, we discuss the linear $\mathbf{G}$ and angular $(\mathbf{H}, \mathbf{H}_O, \mathbf{H}_A)$ momenta of rigid bodies and the inertias that accompany them.[1]

8.1 The Motion of a Rigid Body

8.1.1 General Considerations

A body $\mathscr{B}$ is a collection of material points (mass particles or particles). We denote a material point by X. The position of the material point X, relative to a fixed origin, at time t is denoted by $\mathbf{x}$ (see Figure 8.1). The present (or current) configuration κ_t of the body is a smooth, one-to-one, onto function that has a continuous inverse. It maps material points X of $\mathscr{B}$ to points in three-dimensional Euclidean space: $\mathbf{x} = \kappa_t(X)$. As the location $\mathbf{x}$ of the particle X changes with time, this function depends on time, hence the subscript t.

It is convenient to define a fixed reference configuration κ_0 of the body. This configuration is defined by the function $\mathbf{X} = \kappa_0(X)$. Because this function is invertible,

[1] The details presented here are far more advanced than those in most undergraduate texts. This is partially because the presentation is influenced by the recent renaissance in continuum mechanics. We mention in particular the influential works by Beatty [5] and Casey [13, 14, 15, 16], who used the fruits of this era to present enlightening treatments of rigid body mechanics. This chapter is based on the aforementioned works and Chapter 4 of Gurtin [33].

O.M. O'Reilly, *Engineering Dynamics: A Primer*, DOI 10.1007/978-1-4419-6360-4_8,

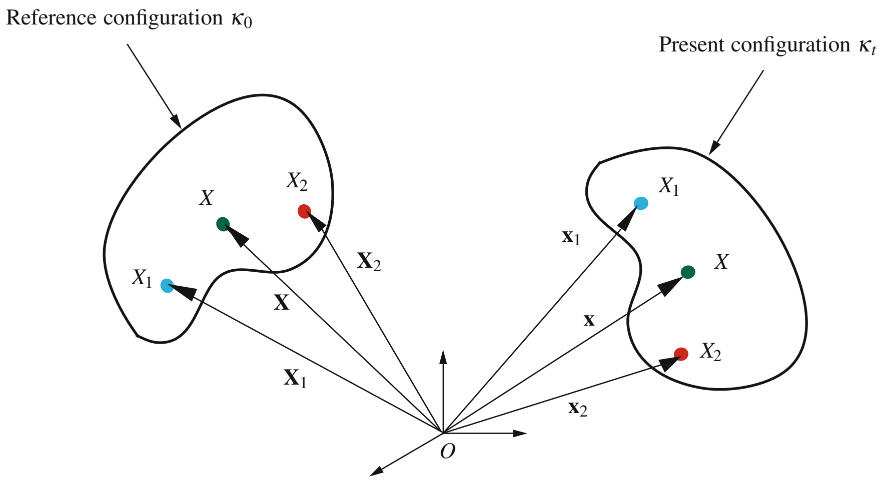

Fig. 8.1 *Configurations of a body* $\mathscr{B}$.

we can use the position vector $\mathbf{X}$ of a material point X in the reference configuration to uniquely define the material point of interest. One can then define the motion of the body as a function of $\mathbf{X}$ and t:

$$\mathbf{x} = \chi(\mathbf{X}, t).$$

Notice that the motion of a material point of $\mathscr{B}$ depends on time and the material point of interest. To see this imagine a compact disc spinning in a CD player. The motion of a particle on the outer rim of the disc is clearly different from the motion of a particle on the inner rim of the disc. Furthermore, the place in space that each of these particles occupies depends on the time t of interest.

The previous developments are general and are used in continuum mechanics, a field that encompasses the mechanics of solids and fluids.

8.1.2 Rigidity

For rigid bodies, the nature of the function $\chi(\mathbf{X}, t)$ can be simplified dramatically. We refer to the rigid motion as $\mathbf{x} = \chi_R(\mathbf{X}, t)$. First, for rigid bodies the distance between *any* two mass particles, say X_1 and X_2, remains constant for all motions. Mathematically, this is equivalent to saying that

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\mathbf{X}_1 - \mathbf{X}_2\|.$$

Secondly, the motion of the rigid body preserves orientations. Using a classical result,[2] it can be proven that the motion of a rigid body has the form

$$\begin{bmatrix} (\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathbf{E}_x \\ (\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathbf{E}_y \\ (\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathbf{E}_z \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} (\mathbf{X}_1 - \mathbf{X}_2) \cdot \mathbf{E}_x \\ (\mathbf{X}_1 - \mathbf{X}_2) \cdot \mathbf{E}_y \\ (\mathbf{X}_1 - \mathbf{X}_2) \cdot \mathbf{E}_z \end{bmatrix}.$$

Here, the matrix whose components are $Q_{11}, \ldots, Q_{33}$ is a proper-orthogonal or rotation matrix.

To abbreviate the subsequent developments, we introduce notations for a matrix, its transpose, and the identity matrix:

$$\mathsf{Q} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix}, \qquad \mathsf{Q}^T = \begin{bmatrix} Q_{11}(t) & Q_{21}(t) & Q_{31}(t) \\ Q_{12}(t) & Q_{22}(t) & Q_{32}(t) \\ Q_{13}(t) & Q_{23}(t) & Q_{33}(t) \end{bmatrix},$$

$$\mathsf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The relations satisfied by any rotation matrix Q can be expressed compactly as

$$\mathsf{Q}\mathsf{Q}^T = \mathsf{I}, \qquad \det(\mathsf{Q}) = 1.$$

In words, the inverse of a rotation matrix is its transpose, and the determinant of a rotation matrix is one. The relations $\mathsf{Q}\mathsf{Q}^T = \mathsf{I}$ can be interpreted as six conditions on the nine components of Q. It follows that only three of the nine components Q_{ik} are independent. One then has the problem of parametrizing Q in terms of three independent parameters. There are several methods of doing this, Euler angles being the most popular.[3] In an undergraduate engineering dynamics course one considers a particular one-parameter family of rotation matrices. Finally, we note that $\det(\mathsf{Q}) = 1$ implies that Q preserves orientations.

Because Q is a rotation matrix, we note that[4]

$$0 = \frac{d\mathsf{I}}{dt} = \frac{d(\mathsf{Q}\mathsf{Q}^T)}{dt} = \frac{d\mathsf{Q}}{dt}\mathsf{Q}^T + \mathsf{Q}\frac{d\mathsf{Q}^T}{dt}.$$

[2] The proof of this result is beyond the scope of this course. One proof may be found on pages 49–50 of Gurtin [33]. A good discussion on the relationship between this result with Euler's theorem on the motion of a rigid body and Chasles' theorem can be found in Beatty [5] (see also Beatty [4]). Euler's representation of rigid body motion can be seen on pages 30–32 of Euler [26].

[3] Details on these parametrizations can be found, for example, in Beatty [5], Greenwood [32], O'Reilly [55], Shuster [71], Synge and Griffith [78], and Whittaker [81].

[4] Recall that the transpose of a product of two matrices A and B is $(\mathsf{A}\mathsf{B})^T = \mathsf{B}^T\mathsf{A}^T$. Furthermore, a matrix C is symmetric if $\mathsf{C} = \mathsf{C}^T$ and is skew-symmetric if $\mathsf{C} = -\mathsf{C}^T$.

That is,

$$\frac{d\mathsf{Q}}{dt}\mathsf{Q}^T = -\mathsf{Q}\frac{d\mathsf{Q}^T}{dt} = -\left(\frac{d\mathsf{Q}}{dt}\mathsf{Q}^T\right)^T.$$

Hence, $\dot{\mathsf{Q}}\mathsf{Q}^T$ is a skew-symmetric matrix:

$$\dot{\mathsf{Q}}\mathsf{Q}^T = \begin{bmatrix} 0 & -\Omega_{21} & \Omega_{13} \\ \Omega_{21} & 0 & -\Omega_{32} \\ -\Omega_{13} & \Omega_{32} & 0 \end{bmatrix}.$$

The three components Ω_{21}, Ω_{13}, and Ω_{32} can be expressed in terms of the components of Q and its time derivatives, but we leave this as an exercise.

8.1.3 Angular Velocity and Acceleration Vectors

Returning to the discussion of a rigid body, we recall that

$$\begin{bmatrix} (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_x \\ (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_y \\ (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_x \\ (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_y \\ (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_z \end{bmatrix}.$$

Because the matrix Q is a rotation matrix, we can easily invert this relationship by multiplying both sides of it by the transpose of Q:

$$\begin{bmatrix} (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_x \\ (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_y \\ (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{21}(t) & Q_{31}(t) \\ Q_{12}(t) & Q_{22}(t) & Q_{32}(t) \\ Q_{13}(t) & Q_{23}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_x \\ (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_y \\ (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_z \end{bmatrix}.$$

Now let us examine the relationship between the velocity and acceleration vectors of two material points of the body. A simple differentiation, where we note that $\mathbf{X}_1$ and $\mathbf{X}_2$ are constant, gives

$$\begin{bmatrix} (\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{E}_x \\ (\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{E}_y \\ (\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \dot{Q}_{11}(t) & \dot{Q}_{12}(t) & \dot{Q}_{13}(t) \\ \dot{Q}_{21}(t) & \dot{Q}_{22}(t) & \dot{Q}_{23}(t) \\ \dot{Q}_{31}(t) & \dot{Q}_{32}(t) & \dot{Q}_{33}(t) \end{bmatrix} \begin{bmatrix} (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_x \\ (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_y \\ (\mathbf{X}_1 - \mathbf{X}_2)\cdot\mathbf{E}_z \end{bmatrix}.$$

Here, $\mathbf{v}_1 = \dot{\mathbf{x}}_1$ and $\mathbf{v}_2 = \dot{\mathbf{x}}_2$. We next substitute for $\mathbf{X}_1$ and $\mathbf{X}_2$ and use the earlier observation about $\dot{\mathsf{Q}}\mathsf{Q}^T$ to find that

$$\begin{bmatrix} (\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{E}_x \\ (\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{E}_y \\ (\mathbf{v}_1 - \mathbf{v}_2)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_{21} & \Omega_{13} \\ \Omega_{21} & 0 & -\Omega_{32} \\ -\Omega_{13} & \Omega_{32} & 0 \end{bmatrix} \begin{bmatrix} (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_x \\ (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_y \\ (\mathbf{x}_1 - \mathbf{x}_2)\cdot\mathbf{E}_z \end{bmatrix}.$$

We can write this important result in vector notation:

$$\mathbf{v}_1 - \mathbf{v}_2 = \boldsymbol{\omega} \times (\mathbf{x}_1 - \mathbf{x}_2),$$

where $\boldsymbol{\omega}$ is known as the *angular velocity vector* of the rigid body:

$$\boldsymbol{\omega} = \Omega_{32}\mathbf{E}_x + \Omega_{13}\mathbf{E}_y + \Omega_{21}\mathbf{E}_z.$$

You should notice that this vector depends on time and not on the particle of the body: it has the same value for each X. This is because $\boldsymbol{\omega}$ is obtained by differentiating Q and the matrix Q is a function of t only.

We can easily find the relationships between the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ of the material points X_1 and X_2 by differentiating the relationship between their velocities:

$$\begin{aligned}\mathbf{a}_1 - \mathbf{a}_2 &= \dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2 \\ &= \dot{\boldsymbol{\omega}} \times (\mathbf{x}_1 - \mathbf{x}_2) + \boldsymbol{\omega} \times (\mathbf{v}_1 - \mathbf{v}_2) \\ &= \boldsymbol{\alpha} \times (\mathbf{x}_1 - \mathbf{x}_2) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{x}_1 - \mathbf{x}_2)).\end{aligned}$$

Here, $\boldsymbol{\alpha}$ is the *angular acceleration vector* of the rigid body:

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}.$$

8.1.4 Fixed-Axis Rotation

All of the aforementioned developments are general. In an introductory undergraduate engineering dynamics course one considers a special case. In this special case, the axis of rotation is fixed and is normally taken to coincide with $\mathbf{E}_z$.

For this special case, the rotation matrix Q has a particularly simple form:

$$\mathsf{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The angle θ represents a counterclockwise rotation of the rigid body about $\mathbf{E}_z$.

Let us now establish the angular velocity and acceleration vectors associated with this rotation matrix:

$$\begin{aligned}\dot{\mathsf{Q}}\mathsf{Q}^T &= \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \dot{\theta} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

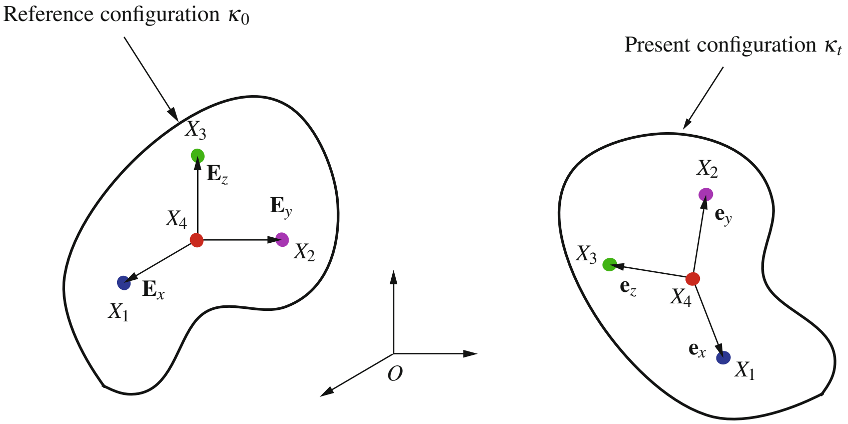

Fig. 8.2 *The corotational basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *and the fixed Cartesian basis* $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$.

Hence, examining the components of the above matrix, we conclude that

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z, \qquad \boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z.$$

8.2 Kinematical Relations and a Corotational Basis

In our previous developments we used a fixed (right-handed) Cartesian basis. It is convenient, when discussing the dynamics of rigid bodies, to introduce another basis which is known as a *corotational* basis.[5] This section discusses such a basis and points out some features of its use.

8.2.1 The Corotational Basis

Here, we define a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ that rotates with the body. As a result, it is known as a corotational basis. Our discussion of this basis follows Casey [13, 16].

Referring to Figure 8.2, we start by picking four material points X_1, X_2, X_3, and X_4 of the body. These points are chosen such that the three vectors

$$\mathbf{E}_x = \mathbf{X}_1 - \mathbf{X}_4, \qquad \mathbf{E}_y = \mathbf{X}_2 - \mathbf{X}_4, \qquad \mathbf{E}_z = \mathbf{X}_3 - \mathbf{X}_4$$

form a fixed, right-handed, Cartesian basis. We next consider the present relative locations of the four material points. Because Q preserves lengths and orientations, the

[5] This basis is often referred to as a body fixed frame or an embedded frame.

three vectors $\mathbf{x}_1 - \mathbf{x}_4$, $\mathbf{x}_2 - \mathbf{x}_4$, and $\mathbf{x}_3 - \mathbf{x}_4$ will also form a right-handed orthonormal basis.[6] As a result, we define the corotational basis to be

$$\mathbf{e}_x = \mathbf{x}_1 - \mathbf{x}_4, \qquad \mathbf{e}_y = \mathbf{x}_2 - \mathbf{x}_4, \qquad \mathbf{e}_z = \mathbf{x}_3 - \mathbf{x}_4.$$

Inasmuch as the corotational basis moves with the body, we can use our previous results for relative velocities in Section 8.1.3[7] to show that

$$\dot{\mathbf{e}}_x = \boldsymbol{\omega} \times \mathbf{e}_x, \qquad \dot{\mathbf{e}}_y = \boldsymbol{\omega} \times \mathbf{e}_y, \qquad \dot{\mathbf{e}}_z = \boldsymbol{\omega} \times \mathbf{e}_z.$$

Furthermore, we can differentiate these results to find that

$$\ddot{\mathbf{e}}_x = \boldsymbol{\alpha} \times \mathbf{e}_x + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}_x).$$

Related results hold for $\ddot{\mathbf{e}}_y$ and $\ddot{\mathbf{e}}_z$. The aforementioned relations prove useful when we establish certain kinematical results later on.

Because the set $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is a basis, given any vector $\mathbf{b}$, one has the representations

$$\begin{aligned}\mathbf{b} &= b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z \\ &= B_x\mathbf{E}_x + B_y\mathbf{E}_y + B_z\mathbf{E}_z.\end{aligned}$$

If $\mathbf{b}$ is a constant vector, $\dot{\mathbf{b}} = \mathbf{0}$, then B_x, B_y, and B_z are constant. However, because the corotational basis changes with time, the constancy of $\mathbf{b}$ does not imply that b_x, b_y, and b_z are constant.

8.2.2 The Corotational Basis for the Fixed-Axis Case

As mentioned previously, for the fixed-axis case, the rotation matrix Q has a particularly simple form:

$$\mathsf{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The angle θ represents a counterclockwise rotation of the rigid body about $\mathbf{E}_z$.

[6] The proof of this result is beyond our scope here. A proof may be found in Casey [13]. For the special case of a fixed-axis rotation, we give an explicit demonstration of this result in Section 8.2.2.

[7] For example, $\dot{\mathbf{e}}_x = \mathbf{v}_1 - \mathbf{v}_4 = \boldsymbol{\omega} \times (\mathbf{x}_1 - \mathbf{x}_4) = \boldsymbol{\omega} \times \mathbf{e}_x$.

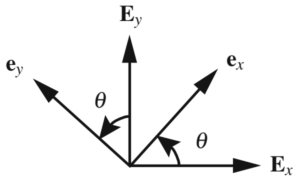

Fig. 8.3 *The corotational basis for a fixed-axis rotation about* $\mathbf{E}_z$.

Taking the aforementioned four points, we find that

$$\begin{bmatrix} (\mathbf{e}_x = \mathbf{x}_1 - \mathbf{x}_4)\cdot\mathbf{E}_x \\ (\mathbf{e}_x = \mathbf{x}_1 - \mathbf{x}_4)\cdot\mathbf{E}_y \\ (\mathbf{e}_x = \mathbf{x}_1 - \mathbf{x}_4)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mathbf{E}_x = \mathbf{X}_1 - \mathbf{X}_4)\cdot\mathbf{E}_x \\ (\mathbf{E}_x = \mathbf{X}_1 - \mathbf{X}_4)\cdot\mathbf{E}_y \\ (\mathbf{E}_x = \mathbf{X}_1 - \mathbf{X}_4)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} (\mathbf{e}_y = \mathbf{x}_2 - \mathbf{x}_4)\cdot\mathbf{E}_x \\ (\mathbf{e}_y = \mathbf{x}_2 - \mathbf{x}_4)\cdot\mathbf{E}_y \\ (\mathbf{e}_y = \mathbf{x}_2 - \mathbf{x}_4)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mathbf{E}_y = \mathbf{X}_2 - \mathbf{X}_4)\cdot\mathbf{E}_x \\ (\mathbf{E}_y = \mathbf{X}_2 - \mathbf{X}_4)\cdot\mathbf{E}_y \\ (\mathbf{E}_y = \mathbf{X}_2 - \mathbf{X}_4)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} (\mathbf{e}_z = \mathbf{x}_3 - \mathbf{x}_4)\cdot\mathbf{E}_x \\ (\mathbf{e}_z = \mathbf{x}_3 - \mathbf{x}_4)\cdot\mathbf{E}_y \\ (\mathbf{e}_z = \mathbf{x}_3 - \mathbf{x}_4)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mathbf{E}_z = \mathbf{X}_3 - \mathbf{X}_4)\cdot\mathbf{E}_x \\ (\mathbf{E}_z = \mathbf{X}_3 - \mathbf{X}_4)\cdot\mathbf{E}_y \\ (\mathbf{E}_z = \mathbf{X}_3 - \mathbf{X}_4)\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In summary, we obtain the results shown graphically in Figure 8.3:

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_x \\ \mathbf{E}_y \\ \mathbf{E}_z \end{bmatrix}.$$

These relations have a familiar form. It is a useful exercise to verify that the corotational basis is right-handed: $\mathbf{e}_z \cdot (\mathbf{e}_x \times \mathbf{e}_y) = 1$.

For the case of a fixed-axis rotation about $\mathbf{E}_z$, we showed previously that

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z, \qquad \boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z.$$

We can use these results to conclude that

$$\begin{aligned} \dot{\mathbf{e}}_x &= \boldsymbol{\omega} \times \mathbf{e}_x = \dot{\theta}\mathbf{e}_y, \\ \dot{\mathbf{e}}_y &= \boldsymbol{\omega} \times \mathbf{e}_y = -\dot{\theta}\mathbf{e}_x, \\ \dot{\mathbf{e}}_z &= \boldsymbol{\omega} \times \mathbf{e}_z = \mathbf{0}. \end{aligned}$$

Alternatively, we can work directly with the representations for $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ in terms of $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ to arrive at the same results.

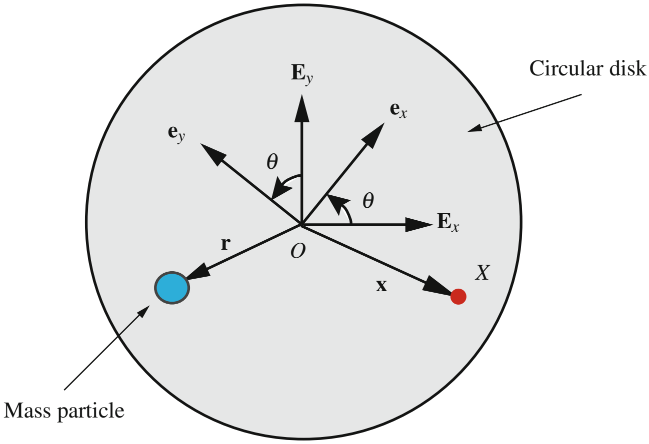

Fig. 8.4 *A particle moving on a rotating disk.*

8.2.3 A Particle Moving on a Rigid Body

It is convenient at this stage to consider an example. As shown in Figure 8.4, a particle moves on the surface of a circular disk. The disk is rotating about the $\mathbf{E}_z$ axis with an angular speed $\dot{\theta} = \omega$ and an angular acceleration $\ddot{\theta} = \alpha$. The center of the disk O is fixed. We seek to determine the velocity vector of the particle and the velocity vector of a point X of the disk.

For the example of interest, suppose that the position vector of the particle is given by the function

$$\mathbf{r} = 10t^2\mathbf{e}_x + 20t\mathbf{e}_y.$$

Furthermore, let the position vector of X be

$$\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y,$$

where x and y are constants.

To calculate the velocity vectors, we merely differentiate the position vectors and use our previous results on the derivatives of $\mathbf{e}_x$ and $\mathbf{e}_y$:

$$\begin{aligned}
\dot{\mathbf{r}} &= 20t\mathbf{e}_x + 10t^2\dot{\mathbf{e}}_x + 20\mathbf{e}_y + 20t\dot{\mathbf{e}}_y \\
&= 20t\mathbf{e}_x + 10t^2\omega\mathbf{e}_y + 20\mathbf{e}_y - 20t\omega\mathbf{e}_x, \\
\dot{\mathbf{x}} &= x\dot{\mathbf{e}}_x + y\dot{\mathbf{e}}_y \\
&= x\omega\mathbf{e}_y - y\omega\mathbf{e}_x.
\end{aligned}$$

You should notice that

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}, \qquad \dot{\mathbf{r}} \neq \boldsymbol{\omega} \times \mathbf{r}.$$

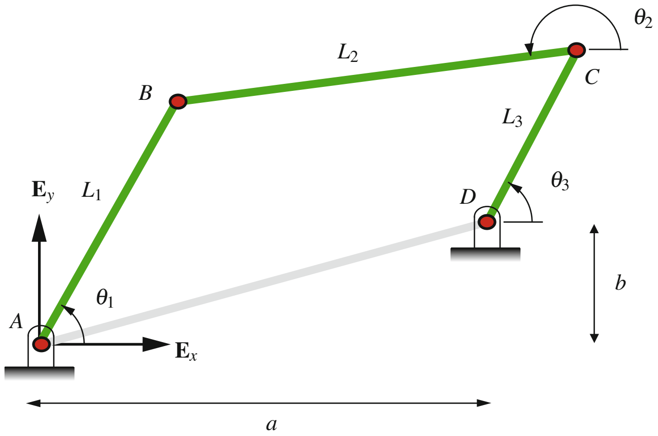

Fig. 8.5 *A four-bar linkage.*

The reason for these results lies in the fact that $\mathbf{x}$ is the position vector of a point of the disk and $\mathbf{r}$ is the position vector of a particle that moves relative to the disk.

We leave it as an exercise to determine the acceleration vectors of the particle and X.

8.3 Mechanisms

One of the main applications of the theory of rigid bodies is an analysis of the kinematics of mechanisms. Two of the most important mechanisms are the slider crank and the four-bar linkage. In general, elements of mechanisms are deformable bodies, but a primitive analysis assumes that these elements are rigid. Here, we also assume that the motions of these elements are coplanar but it is not very difficult to consider the more general case.[8]

As an example, consider the four-bar linkage shown in Figure 8.5. Here, the bar AD is fixed:

$$\mathbf{v}_A = \mathbf{v}_D = \mathbf{0}.$$

The motion of the bar AB is assumed to be known. In other words, θ_1, $\dot{\theta}_1$, and $\ddot{\theta}_1$ are prescribed. The bars AD, DC, AB, and BC are interconnected by pin-joints. One seeks to determine the motion of the two bars DC and BC. That is, one seeks $\theta_2, \dot{\theta}_2, \ddot{\theta}_2, \theta_3, \dot{\theta}_3$, and $\ddot{\theta}_3$ as functions of time. You should note that the angular

[8] The study of mechanisms is an important area of mechanical engineering. Our discussion here touches but a small part of it. The interested reader is referred to the textbooks of Bottema and Roth [11], Mabie and Ocvirk [47], and Paul [59] for further treatments and issues.

velocity vectors of the bars AB, BC, and DC are, respectively,

$$\boldsymbol{\omega}_{AB} = \dot{\theta}_1 \mathbf{E}_z, \qquad \boldsymbol{\omega}_{BC} = \dot{\theta}_2 \mathbf{E}_z, \qquad \boldsymbol{\omega}_{DC} = \dot{\theta}_3 \mathbf{E}_z.$$

First, the linkages are connected together:

$$\mathbf{r}_{DA} = \mathbf{r}_{BA} + \mathbf{r}_{CB} + \mathbf{r}_{DC},$$

where $\mathbf{r}_{DA} = \mathbf{r}_D - \mathbf{r}_A$ and so on. Introducing the angles shown in Figure 8.5, we find that this relationship can be written as

$$\begin{aligned}\mathbf{r}_{DA} = a\mathbf{E}_x + b\mathbf{E}_y &= L_1(\cos(\theta_1)\mathbf{E}_x + \sin(\theta_1)\mathbf{E}_y) \\ &\quad - L_2(\cos(\theta_2)\mathbf{E}_x + \sin(\theta_2)\mathbf{E}_y) - L_3(\cos(\theta_3)\mathbf{E}_x + \sin(\theta_3)\mathbf{E}_y).\end{aligned}$$

This constitutes two scalar equations for the unknown angles θ_2 and θ_3:

$$\begin{aligned}a &= L_1\cos(\theta_1) - L_2\cos(\theta_2) - L_3\cos(\theta_3), \\ b &= L_1\sin(\theta_1) - L_2\sin(\theta_2) - L_3\sin(\theta_3).\end{aligned}$$

These equations are nonlinear and, in general, have multiple solutions (θ_2, θ_3). To see this, one merely has to draw different possible configurations of the mechanism.

To obtain a second set of equations, we differentiate the position vector relationship above:

$$\mathbf{v}_{DA} = \mathbf{v}_{BA} + \mathbf{v}_{CB} + \mathbf{v}_{DC}.$$

Writing out the two scalar equations, we find that

$$\begin{aligned}0 &= -L_1\dot{\theta}_1\sin(\theta_1) + L_2\dot{\theta}_2\sin(\theta_2) + L_3\dot{\theta}_3\sin(\theta_3), \\ 0 &= L_1\dot{\theta}_1\cos(\theta_1) - L_2\dot{\theta}_2\cos(\theta_2) - L_3\dot{\theta}_3\cos(\theta_3).\end{aligned}$$

To solve these equations for the unknown velocities $\dot{\theta}_2$ and $\dot{\theta}_3$, it is convenient to write them in matrix form:

$$\begin{bmatrix} L_1\dot{\theta}_1\sin(\theta_1) \\ L_1\dot{\theta}_1\cos(\theta_1) \end{bmatrix} = \begin{bmatrix} \sin(\theta_2) & \sin(\theta_3) \\ \cos(\theta_2) & \cos(\theta_3) \end{bmatrix} \begin{bmatrix} L_2\dot{\theta}_2 \\ L_3\dot{\theta}_3 \end{bmatrix}.$$

Inverting the matrix and using a trigonometric identity,[9] the desired results are obtained:

$$\dot{\theta}_2 = \left(\frac{L_1\sin(\theta_1 - \theta_3)}{L_2\sin(\theta_2 - \theta_3)}\right)\dot{\theta}_1, \qquad \dot{\theta}_3 = \left(\frac{L_1\sin(\theta_2 - \theta_1)}{L_3\sin(\theta_2 - \theta_3)}\right)\dot{\theta}_1.$$

[9] The identity that we use is $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$. We also recall the expression for the inverse of a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

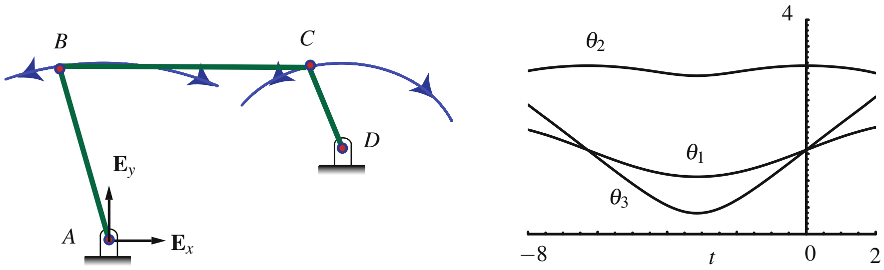

Fig. 8.6 *An example of the motions of a four-bar linkage where $a = L_1 = L_2 = 10$, $L_3 = b = 5$. The image on the left shows the motions of the linkages, and the image on the right displays the evolution of the angles θ_1, θ_2, and θ_3.*

This solution is valid provided that $\sin(\theta_2 - \theta_3)$ is not equal to zero. This occurs when the bars *BC* and *DC* are parallel, and then it is not possible to determine the angular velocities of the bars *DC* and *BC*.

To establish equations to determine the angular accelerations of the bars *DC* and *BC*, we could differentiate the previous velocity vector equation to obtain

$$\mathbf{a}_{DA} = \mathbf{a}_{BA} + \mathbf{a}_{CB} + \mathbf{a}_{DC}.$$

The resulting two scalar equations, when supplemented by the two scalar position equations and two scalar velocity equations, could be used to obtain expressions for $\ddot{\theta}_2$ and $\ddot{\theta}_3$. An easier method of obtaining the desired accelerations is to differentiate the previous expressions for $\dot{\theta}_2$ and $\dot{\theta}_3$:

$$\begin{aligned}
\ddot{\theta}_2 &= \frac{d}{dt}\left(\frac{L_1 \sin(\theta_1 - \theta_3)}{L_2 \sin(\theta_2 - \theta_3)}\right)\dot{\theta}_1 + \left(\frac{L_1 \sin(\theta_1 - \theta_3)}{L_2 \sin(\theta_2 - \theta_3)}\right)\ddot{\theta}_1, \\
\ddot{\theta}_3 &= \frac{d}{dt}\left(\frac{L_1 \sin(\theta_2 - \theta_1)}{L_3 \sin(\theta_2 - \theta_3)}\right)\dot{\theta}_1 + \left(\frac{L_1 \sin(\theta_2 - \theta_1)}{L_3 \sin(\theta_2 - \theta_3)}\right)\ddot{\theta}_1.
\end{aligned}$$

In conclusion, given the motion of the link *AB*, it is possible to solve for the angular displacements, angular speeds, and angular accelerations of the bars *DC* and *BC*. To illustrate these developments, we consider a specific mechanism and use the results established in this section to determine θ_2 and θ_3 for a given $\theta_1(t) = \pi/2 + 0.5\sin(0.5t)$. The results are shown in Figure 8.6, and can be used to infer the angular velocities and accelerations of the links.

We note that the analysis of a slider crank mechanism is similar to that presented here for a four-bar linkage. For the slider crank, the six unknowns are the displacement of the slider and the angular displacement of the link connecting the slider to the crank, along with the first and second time derivatives of these quantities.

8.4 Center of Mass and Linear Momentum

In all of the previous developments, we defined the motion of one material point relative to another material point of the same body. It is convenient for later purposes to now define a particular point: the center of mass C.

We first dispense with some preliminaries. Let $\mathscr{R}_0$ and $\mathscr{R}$ denote the regions of Euclidean three-space occupied by the body in its reference and present configurations, respectively. Furthermore, let $\mathbf{X}$ and $\mathbf{x}$ be the position vectors of a material point X of the body in its reference and present configurations, respectively (see Figure 8.1).

8.4.1 *The Center of Mass*

The position vectors of the center of mass of the body in its reference and present configurations are defined by

$$\bar{\mathbf{X}} = \frac{\int_{\mathscr{R}_0} \mathbf{X}\rho_0 dV}{\int_{\mathscr{R}_0} \rho_0 dV}, \qquad \bar{\mathbf{x}} = \frac{\int_{\mathscr{R}} \mathbf{x}\rho dv}{\int_{\mathscr{R}} \rho dv},$$

where $\rho_0 = \rho_0(\mathbf{X})$ and $\rho = \rho(\mathbf{x}, t)$ are the mass densities per unit volume of the body in the reference and present configurations.[10]

We assume that the mass of the body is conserved:

$$dm = \rho_0 dV = \rho dv,$$

$$m = \int_{\mathscr{R}_0} \rho_0 dV = \int_{\mathscr{R}} \rho dv.$$

This is the principle of mass conservation. Hence,

$$m\bar{\mathbf{X}} = \int_{\mathscr{R}_0} \mathbf{X}\rho_0 dV, \qquad m\bar{\mathbf{x}} = \int_{\mathscr{R}} \mathbf{x}\rho dv.$$

In addition, one has the useful identities

$$\mathbf{0} = \int_{\mathscr{R}_0} (\mathbf{X} - \bar{\mathbf{X}})\rho_0 dV, \qquad \mathbf{0} = \int_{\mathscr{R}} (\mathbf{x} - \bar{\mathbf{x}})\rho dv.$$

You should compare these expressions to those we obtained in Chapter 7 for a system of particles.

A special feature of rigid bodies is that the center of mass behaves as if it were a material point. We denote this point by C. For many bodies, such as a rigid

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[10] If a body is homogeneous, then ρ_0 is a constant that is independent of $\mathbf{X}$. Our use of the symbol ρ here should not be confused with our use of the same symbol for the radius of curvature of a space curve in Chapter 3.

homogeneous sphere, the center of mass corresponds to the geometric center of the sphere, whereas for others, such as a rigid circular ring, it does not correspond to a material point. It can be proven that for any material point Y of a rigid body, one has[11]

$$\begin{bmatrix} (\bar{\mathbf{x}}-\mathbf{y})\cdot\mathbf{E}_x \\ (\bar{\mathbf{x}}-\mathbf{y})\cdot\mathbf{E}_y \\ (\bar{\mathbf{x}}-\mathbf{y})\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} (\bar{\mathbf{X}}-\mathbf{Y})\cdot\mathbf{E}_x \\ (\bar{\mathbf{X}}-\mathbf{Y})\cdot\mathbf{E}_y \\ (\bar{\mathbf{X}}-\mathbf{Y})\cdot\mathbf{E}_z \end{bmatrix}.$$

That is, $\bar{\mathbf{x}} = \chi_R(\bar{\mathbf{X}},t)$. Differentiating these results as in Section 8.1.3, we find that

$$\begin{aligned} \bar{\mathbf{v}} - \dot{\mathbf{y}} &= \boldsymbol{\omega} \times (\bar{\mathbf{x}}-\mathbf{y}), \\ \bar{\mathbf{a}} - \ddot{\mathbf{y}} &= \boldsymbol{\alpha} \times (\bar{\mathbf{x}}-\mathbf{y}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\bar{\mathbf{x}}-\mathbf{y})). \end{aligned}$$

Here, $\bar{\mathbf{v}}$ and $\bar{\mathbf{a}}$ are the velocity and acceleration vectors of the center of mass C.

8.4.2 The Linear Momentum

We next turn to the linear momentum $\mathbf{G}$ of a rigid body. By definition, this momentum is

$$\mathbf{G} = \int_{\mathscr{R}} \mathbf{v}\rho dv.$$

That is, the linear momentum of a rigid body is the sum of the linear momenta of its constituents. We can establish an alternative expression for $\mathbf{G}$ using the center of mass:[12]

$$\begin{aligned} \mathbf{G} = \int_{\mathscr{R}} \mathbf{v}\rho dv &= \int_{\mathscr{R}} \frac{d\mathbf{x}}{dt}\rho dv \\ &= \frac{d}{dt}\left(\int_{\mathscr{R}} \mathbf{x}\rho dv\right) \\ &= \frac{d}{dt}(m\bar{\mathbf{x}}). \end{aligned}$$

Hence,

$$\mathbf{G} = m\bar{\mathbf{v}}.$$

You may recall that a related result holds for a system of particles.

[11] The proof is beyond the scope of an undergraduate engineering dynamics course. For completeness, however, one proof is presented in Section 8.9.

[12] Some may notice that we take the time derivative to the outside of an integral whose region of integration $\mathscr{R}$ depends on time. This is generally not possible. However, for the integral of interest it is shown in Section 8.9 that such a manipulation is justified.

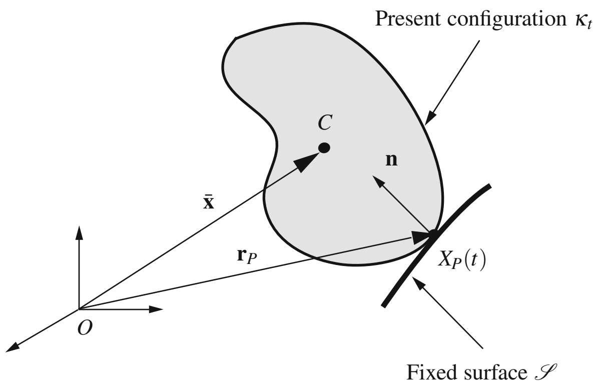

Fig. 8.7 *The geometry of contact.*

8.5 Kinematics of Rolling and Sliding

Many mechanical systems consist of a body in motion that is in contact with one point of another body. The resulting contact conditions are known as kinematical constraints. The discussion and description of these constraints is complicated by the fact that the particular material point of the body that is in contact changes with time.

There are two types of contact that are of interest here: rolling and sliding. The study of rolling and sliding contact is a classical area of dynamics. In particular, some rolling rigid bodies such as the wobblestone (also known as a celt or rattle-back) exhibit interesting counterintuitive dynamics.[13]

To proceed, we consider a rigid body $\mathscr{B}$ that is in contact with a fixed surface $\mathscr{S}$ (see Figure 8.7). As the body moves on this fixed surface, the material point of the body that is in contact with the surface may change. We denote the material point of the body that is in contact at time t by $P = X_P(t)$. We denote the position vector of P by $\mathbf{r}_P$ and its velocity vector by $\mathbf{v}_P$. Finally, the unit normal to $\mathscr{S}$ at P is denoted by $\mathbf{n}$.

For any material point X of $\mathscr{B}$, recall that the velocity and acceleration vectors are

$$\begin{aligned}
\mathbf{v} &= \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{x} - \bar{\mathbf{x}}), \\
\mathbf{a} &= \bar{\mathbf{a}} + \boldsymbol{\alpha} \times (\mathbf{x} - \bar{\mathbf{x}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{x} - \bar{\mathbf{x}})).
\end{aligned}$$

[13] The standard modern reference to this area was written by two Soviet mechanicians: Neimark and Fufaev [51]. One of the prime contributors to this area was Routh [64, 65]. Indeed, the problem of determining the motion of a sphere rolling on a surface of revolution is known as Routh's problem. We also mention the interesting classical work on billiards (pool) by Coriolis [17] from 1835.

Consequently, for P, one has the relations

$$\mathbf{v}_P = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}}),$$
$$\mathbf{a}_P = \bar{\mathbf{a}} + \boldsymbol{\alpha} \times (\mathbf{r}_P - \bar{\mathbf{x}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}})).$$

For a rigid body that is sliding on the fixed surface $\mathscr{S}$, the component of $\mathbf{v}_P$ in the direction of $\mathbf{n}$ is zero:

$$\mathbf{v}_P \cdot \mathbf{n} = 0.$$

This implies the *sliding condition*:

$$\bar{\mathbf{v}} \cdot \mathbf{n} = -(\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}})) \cdot \mathbf{n}.$$

For a rigid body that is rolling on the fixed surface $\mathscr{S}$, the velocity of the instantaneous point of contact P is zero:

$$\mathbf{v}_P = \mathbf{0}.$$

This implies the *rolling condition*:

$$\bar{\mathbf{v}} = -\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}}).$$

We also note for a rolling rigid body that

$$\mathbf{a}_P = \bar{\mathbf{a}} + \boldsymbol{\alpha} \times (\mathbf{r}_P - \bar{\mathbf{x}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}})),$$

and this acceleration vector is not necessarily $\mathbf{0}$.

8.6 Kinematics of a Rolling Circular Disk

The main examples of rolling rigid bodies are upright rolling disks and cylinders. These examples are often used as simple models for wheel-road interactions in vehicle dynamics as well as numerous examples of bearing surfaces and mechanism driving devices. Here, we focus on a circular disk. The developments for a cylinder are easily inferred.[14]

As shown in Figure 8.8, we consider an upright homogeneous disk of radius R that is rolling on a plane. To start, we define a corotational basis for the disk:

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_y = \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x, \qquad \mathbf{e}_z = \mathbf{E}_z.$$

[14] For further references to, and discussions of, rolling disks and sliding disks see Borisov and Mamaev [10], Cushman et al. [20], Hermans [34], and O'Reilly [54]. References [10, 54] contain discussion of the important works on these systems by Chaplygin in 1897, Appell and Korteweg in 1900, and Vierkandt in 1892.

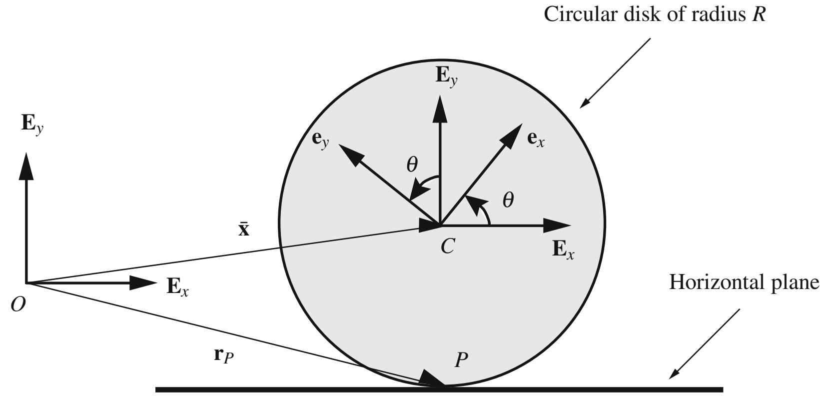

Fig. 8.8 *A circular disk rolling on a horizontal plane.*

Because the motion is a fixed-axis rotation,

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z, \qquad \boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z.$$

Furthermore, because the center of mass C of the disk is at its geometric center,

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z.$$

The position vector of the instantaneous point of contact P of the disk with the plane is

$$\mathbf{r}_P = \bar{\mathbf{x}} - R\mathbf{E}_y.$$

The unit normal $\mathbf{n}$ mentioned earlier is $\mathbf{E}_y$ for this problem.

Because the disk is rolling, $\mathbf{v}_P = \mathbf{0}$, we find that

$$\begin{aligned}\bar{\mathbf{v}} &= -\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}}) = -\dot{\theta}\mathbf{E}_z \times (-R\mathbf{E}_y) \\ &= -R\dot{\theta}\mathbf{E}_x.\end{aligned}$$

This vector equation is equivalent to three scalar equations:

$$\dot{x} = -R\dot{\theta}, \qquad \dot{y} = 0, \qquad \dot{z} = 0.$$

The last two of these equations imply that the velocity of C is only in the $\mathbf{E}_x$ direction, as expected. It follows from these equations that the motion of the center of mass is completely determined by the rotational motion of the disk:

$$\bar{\mathbf{v}} = \dot{x}\mathbf{E}_x = -R\dot{\theta}\mathbf{E}_x, \qquad \bar{\mathbf{a}} = \ddot{x}\mathbf{E}_x = -R\ddot{\theta}\mathbf{E}_x.$$

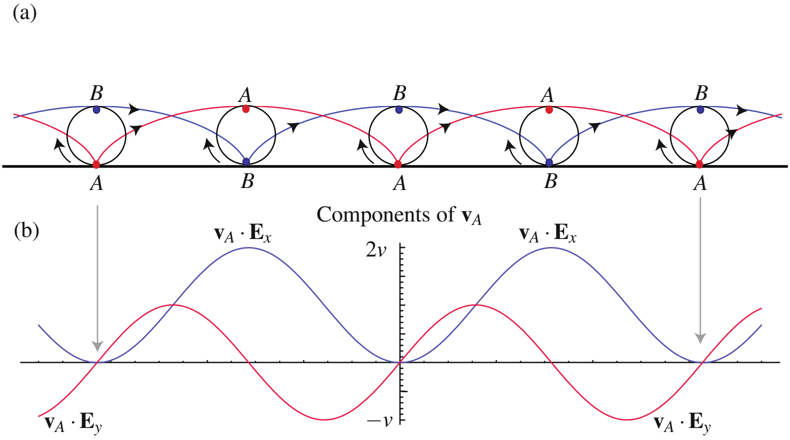

Fig. 8.9 *(a) The loci of two points A and B on a disk that is rolling with a constant angular velocity* $\dot{\theta} < 0$ *on a horizontal surface:* $\bar{\mathbf{v}} = v\mathbf{E}_x = -R\dot{\theta}\mathbf{E}_x$. *In (b) the* $\mathbf{E}_x$ *and* $\mathbf{E}_y$ *components of the velocity vector* $\mathbf{v}_A$ *of the point A on the disk are shown.*

Let us now examine the acceleration of the instantaneous point of contact $P = X_P(t)$. We know that the velocity vector of this point is $\mathbf{0}$. However,

$$\begin{aligned}\mathbf{a}_P &= \bar{\mathbf{a}} + \boldsymbol{\alpha} \times (\mathbf{r}_P - \bar{\mathbf{x}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}})) \\ &= \ddot{x}\mathbf{E}_x + \ddot{\theta}\mathbf{E}_z \times (-R\mathbf{E}_y) + \dot{\theta}\mathbf{E}_z \times (\dot{\theta}\mathbf{E}_z \times -R\mathbf{E}_y) \\ &= R\dot{\theta}^2\mathbf{E}_y.\end{aligned}$$

This acceleration is not zero because the material point X_P that is in contact with the surface changes with time.

To determine the velocity and acceleration vectors of any material point X of the rolling rigid disk, we note that

$$\mathbf{x} - \bar{\mathbf{x}} = x_1\mathbf{e}_x + y_1\mathbf{e}_y,$$

where x_1 and y_1 are constants. Next, we use the previous results

$$\begin{aligned}\mathbf{v} &= \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{x} - \bar{\mathbf{x}}), \\ \mathbf{a} &= \bar{\mathbf{a}} + \boldsymbol{\alpha} \times (\mathbf{x} - \bar{\mathbf{x}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{x} - \bar{\mathbf{x}})).\end{aligned}$$

Because,

$$\bar{\mathbf{v}} = -R\dot{\theta}\mathbf{E}_x, \qquad \bar{\mathbf{a}} = -R\ddot{\theta}\mathbf{E}_x, \qquad \boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z, \qquad \boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z,$$

we find that

$$\begin{aligned}
\mathbf{v} &= -R\dot{\theta}\mathbf{E}_x + \dot{\theta}\mathbf{E}_z \times (x_1\mathbf{e}_x + y_1\mathbf{e}_y) \\
&= -R\dot{\theta}\mathbf{E}_x + \dot{\theta}(x_1\mathbf{e}_y - y_1\mathbf{e}_x), \\
\mathbf{a} &= -R\ddot{\theta}\mathbf{E}_x + \ddot{\theta}\mathbf{E}_z \times (x_1\mathbf{e}_x + y_1\mathbf{e}_y) + \dot{\theta}\mathbf{E}_z \times (\dot{\theta}\mathbf{E}_z \times (x_1\mathbf{e}_x + y_1\mathbf{e}_y)) \\
&= -R\ddot{\theta}\mathbf{E}_x + \ddot{\theta}(x_1\mathbf{e}_y - y_1\mathbf{e}_x) - \dot{\theta}^2(x_1\mathbf{e}_x + y_1\mathbf{e}_y).
\end{aligned}$$

It is a good exercise to choose various values of x_1 and y_1 and examine the corresponding velocity and acceleration vectors.

As an example of the previous exercise, we consider a disk rolling on a horizontal surface at constant speed. The case $x_1 = 0$ and $y_1 = R$ corresponds to the point B in Figure 8.9(a) and the case $x_1 = 0$ and $y_1 = -R$ corresponds to the point A in Figure 8.9(a). In Figure 8.9(b), the components of the velocity vector of the point A are shown. You should notice how $\mathbf{v}_A = \mathbf{0}$ when A corresponds to the instantaneous point of contact. The curves traced by the points A and B are well-known examples of a plane curve known as the cycloid.

8.6.1 A Common Error

It is a commmon error to start with the equation $\mathbf{r}_P = \bar{\mathbf{x}} - R\mathbf{E}_y$, and then differentiate this equation to try to get $\mathbf{v}_P$ and $\mathbf{a}_P$. This leads to the incorrect answers $\mathbf{v}_P = \bar{\mathbf{v}}$ and $\mathbf{a}_P = \bar{\mathbf{a}}$. The reason for these errors lies in the fact that the position vector of $X_P(t)$ relative to the center of mass C is $-R\mathbf{E}_y$ *only* at the instant t. At other times, its relative position vector does not have this value. When one differentiates $\mathbf{r}_P = \bar{\mathbf{x}} - R\mathbf{E}_y$, the derivative of $R\mathbf{E}_y$ is equal to $\mathbf{0}$. Hence, one is falsely assuming that the same material point is the instantaneous point of contact for the entire duration of the motion.

8.6.2 The Sliding Disk

It is interesting to pause briefly to consider the sliding disk. For such a disk, the sliding condition yields

$$\bar{\mathbf{v}} \cdot \mathbf{E}_y = -\left(\dot{\theta}\mathbf{E}_z \times (-R\mathbf{E}_y)\right) \cdot \mathbf{E}_y.$$

This implies that

$$\bar{\mathbf{v}} \cdot \mathbf{E}_y = \dot{y} = 0.$$

Hence, the rotational and translational motions of the disk are not coupled.

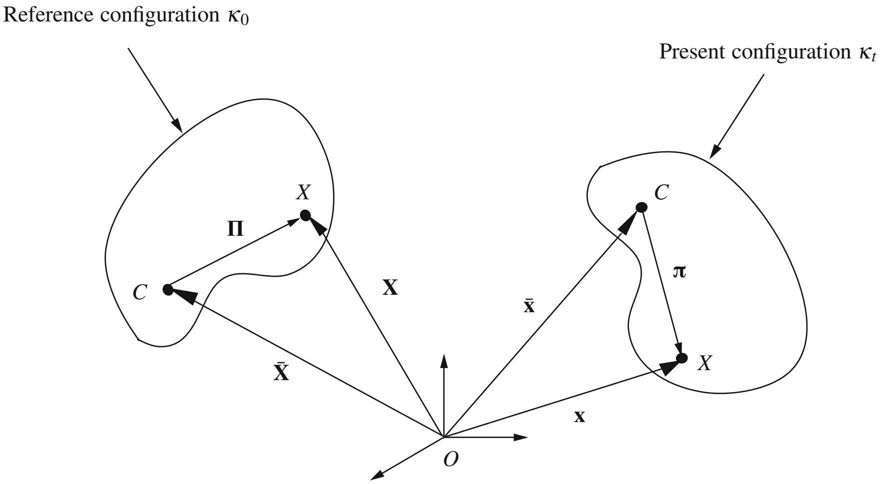

Fig. 8.10 *Relative position vectors* $\mathbf{\Pi}$ *and* $\boldsymbol{\pi}$ *of a material point* X *of a body.*

8.7 Angular Momenta

Preparatory to a discussion of the balance laws for a rigid body, we now address the angular momenta of a rigid body. The momentum relative to two points, the center of mass C and a fixed point O, are of considerable importance in the next two chapters. For convenience, we assume that the fixed point O is also the origin (see Figure 8.10).

By definition, the angular momenta of a rigid body relative to its center of mass C, $\mathbf{H}$, and a fixed point O, $\mathbf{H}_O$, are

$$\mathbf{H} = \int_{\mathscr{R}} (\mathbf{x} - \bar{\mathbf{x}}) \times \mathbf{v} \rho dv, \qquad \mathbf{H}_O = \int_{\mathscr{R}} \mathbf{x} \times \mathbf{v} \rho dv.$$

These momenta are related by a simple and important formula. To find this formula, we perform some manipulations on $\mathbf{H}_O$:

$$\begin{aligned}
\mathbf{H}_O &= \int_{\mathscr{R}} \mathbf{x} \times \mathbf{v} \rho dv \\
&= \int_{\mathscr{R}} (\mathbf{x} - \bar{\mathbf{x}} + \bar{\mathbf{x}}) \times \mathbf{v} \rho dv \\
&= \int_{\mathscr{R}} (\mathbf{x} - \bar{\mathbf{x}}) \times \mathbf{v} \rho dv + \int_{\mathscr{R}} \bar{\mathbf{x}} \times \mathbf{v} \rho dv \\
&= \mathbf{H} + \bar{\mathbf{x}} \times \int_{\mathscr{R}} \mathbf{v} \rho dv.
\end{aligned}$$

That is,

$$\mathbf{H}_O = \mathbf{H} + \bar{\mathbf{x}} \times \mathbf{G}.$$

In words, the angular momentum of a rigid body relative to a fixed point O is the sum of the angular momentum of the rigid body about its center of mass and the angular momentum of its center of mass relative to O.

As an extension of the previous result, it is a good exercise to show that the angular momentum of a rigid body relative to an arbitrary point A satisfies the identity

$$\mathbf{H}_A = \int_{\mathscr{R}} (\mathbf{x} - \mathbf{x}_A) \times \mathbf{v} \rho dv = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_A) \times \mathbf{G}.$$

Here, $\mathbf{x}_A$ is the position vector of the point A. You might recall that we had a similar identity for a system of particles.

In the forthcoming balance laws, we need to measure $\mathbf{H}$ and $\mathbf{H}_O$ at various instants of time. Using the formulae above, this is a tedious task. It is simplified tremendously by the introduction of inertia tensors.

8.8 Inertia Tensors

In this section we first establish the inertia tensor for a rigid body relative to its center of mass C. Some comments on the parallel-axis theorem are presented at the end of this section.

To start, we recall that for any material point X of a rigid body, one has the relation

$$\mathbf{v} = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{x} - \bar{\mathbf{x}}).$$

As shown in Figure 8.10, we also define the relative position vectors

$$\boldsymbol{\pi} = \mathbf{x} - \bar{\mathbf{x}}, \qquad \boldsymbol{\Pi} = \mathbf{X} - \bar{\mathbf{X}}.$$

Because the motion of the body is rigid, these vectors have interesting representations:

$$\begin{aligned} \boldsymbol{\pi} &= \mathbf{x} - \bar{\mathbf{x}} = \Pi_x \mathbf{e}_x + \Pi_y \mathbf{e}_y + \Pi_z \mathbf{e}_z, \\ \boldsymbol{\Pi} &= \mathbf{X} - \bar{\mathbf{X}} = \Pi_x \mathbf{E}_x + \Pi_y \mathbf{E}_y + \Pi_z \mathbf{E}_z. \end{aligned}$$

Notice that the components of $\boldsymbol{\Pi}$ relative to the fixed Cartesian basis are identical to those of $\boldsymbol{\pi}$ relative to the corotational basis. This implies that the latter components can be measured using the fixed reference configuration.

8.8.1 Where the Inertia Tensor Comes From

Consider the angular momentum $\mathbf{H}$:

$$\begin{aligned}\mathbf{H} &= \int_{\mathscr{R}} (\mathbf{x} - \bar{\mathbf{x}}) \times \mathbf{v}\rho dv \\ &= \int_{\mathscr{R}} \boldsymbol{\pi} \times \mathbf{v}\rho dv \\ &= \int_{\mathscr{R}} \boldsymbol{\pi} \times (\bar{\mathbf{v}} + \boldsymbol{\omega} \times \boldsymbol{\pi})\rho dv \\ &= \int_{\mathscr{R}} \boldsymbol{\pi} \times \bar{\mathbf{v}}\rho dv + \int_{\mathscr{R}} \boldsymbol{\pi} \times (\boldsymbol{\omega} \times \boldsymbol{\pi})\rho dv.\end{aligned}$$

However, inasmuch as C is the center of mass and the velocity vector of C is independent of the region of integration,

$$\int_{\mathscr{R}} \boldsymbol{\pi} \times \bar{\mathbf{v}}\rho dv = \int_{\mathscr{R}} \boldsymbol{\pi}\rho dv \times \bar{\mathbf{v}} = \mathbf{0} \times \bar{\mathbf{v}} = \mathbf{0}.$$

Hence,

$$\mathbf{H} = \int_{\mathscr{R}} \boldsymbol{\pi} \times (\boldsymbol{\omega} \times \boldsymbol{\pi})\rho dv = \int_{\mathscr{R}} \left((\boldsymbol{\pi} \cdot \boldsymbol{\pi})\boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega})\boldsymbol{\pi}\right) \rho dv.$$

In writing this equation, we used the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Substituting the representations[15]

$$\begin{aligned}\boldsymbol{\pi} &= \mathbf{x} - \bar{\mathbf{x}} = \Pi_x \mathbf{e}_x + \Pi_y \mathbf{e}_y + \Pi_z \mathbf{e}_z, \\ \boldsymbol{\omega} &= \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z,\end{aligned}$$

into the last equation and expanding, we find that

$$\begin{aligned}\mathbf{H} = &\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)\mathbf{e}_x + \left(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)\mathbf{e}_y \\ &+ \left(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right)\mathbf{e}_z.\end{aligned}$$

In this expression, the inertias $I_{xx}, \ldots, I_{zz}$ are the six independent components of the inertia tensor of the body relative to its center of mass:

$$\begin{aligned}I_{xx} &= \int_{\mathscr{R}} \left(\Pi_y^2 + \Pi_z^2\right) \rho dv = \int_{\mathscr{R}_0} \left(\Pi_y^2 + \Pi_z^2\right) \rho_0 dV, \\ I_{yy} &= \int_{\mathscr{R}} \left(\Pi_x^2 + \Pi_z^2\right) \rho dv = \int_{\mathscr{R}_0} \left(\Pi_x^2 + \Pi_z^2\right) \rho_0 dV, \\ I_{zz} &= \int_{\mathscr{R}} \left(\Pi_x^2 + \Pi_y^2\right) \rho dv = \int_{\mathscr{R}_0} \left(\Pi_x^2 + \Pi_y^2\right) \rho_0 dV,\end{aligned}$$

[15] Earlier, in Section 8.1.3, we expressed ω in terms of the fixed basis: $\boldsymbol{\omega} = \Omega_{32}\mathbf{E}_x + \Omega_{13}\mathbf{E}_y + \Omega_{21}\mathbf{E}_z$. Here, it is more convenient to express $\boldsymbol{\omega}$ in terms of the corotational basis.

and

$$
\begin{aligned}
I_{xy} &= -\int_{\mathscr{R}} \Pi_x \Pi_y \rho dv = -\int_{\mathscr{R}_0} \Pi_x \Pi_y \rho_0 dV, \\
I_{xz} &= -\int_{\mathscr{R}} \Pi_x \Pi_z \rho dv = -\int_{\mathscr{R}_0} \Pi_x \Pi_z \rho_0 dV, \\
I_{yz} &= -\int_{\mathscr{R}} \Pi_y \Pi_z \rho dv = -\int_{\mathscr{R}_0} \Pi_y \Pi_z \rho_0 dV.
\end{aligned}
$$

You should notice that all of the inertias can be evaluated in the fixed reference configuration of the rigid body.[16]

The integrals in the expressions for $I_{xx}, \ldots, I_{zz}$ are standard volume integrals. For many bodies they are tabulated in texts, for example, Table D/4 of Meriam and Kraige [48]. In most texts, I_{xx}, I_{yy}, and I_{zz} are known as the moments of inertia, whereas $-I_{xy}$, $-I_{xz}$, and $-I_{yz}$, are known as the products of inertia.

8.8.2 Angular Momentum and the Inertia Tensor

Recall that we have just shown that

$$
\begin{aligned}
\mathbf{H} = &\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)\mathbf{e}_x + \left(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)\mathbf{e}_y \\
&+ \left(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right)\mathbf{e}_z.
\end{aligned}
$$

We can write this result in a more transparent form:

$$
\begin{bmatrix} \mathbf{H}\cdot\mathbf{e}_x \\ \mathbf{H}\cdot\mathbf{e}_y \\ \mathbf{H}\cdot\mathbf{e}_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}\cdot\mathbf{e}_x \\ \boldsymbol{\omega}\cdot\mathbf{e}_y \\ \boldsymbol{\omega}\cdot\mathbf{e}_z \end{bmatrix}.
$$

The matrix in this equation is known as the inertia matrix. Its components are the components of the inertia tensor.

We notice that the inertia matrix is symmetric. It may also be shown that it is positive definite. As a result, its eigenvalues (or principal values) are real positive numbers.[17] You should also notice that the components of this matrix depend on the corotational basis chosen and, as a result, the fixed Cartesian basis also. If the vectors $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ are chosen to coincide with the eigenvectors of this matrix, then the sets of vectors $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are said to be the principal axes of the body in its reference and present configurations, respectively. In this case, the

[16] These results are discussed in Casey [13, 16] and in Section 13, Chapter 4 of Gurtin [33]. For further details on the transformation of the integrals, see Section 8.9 below.

[17] These results can be easily found in texts on linear algebra; see, e.g., Bellman [8] or Strang [75]. One also uses these results for the (Cauchy) stress tensor when constructing Mohr's circle in solid mechanics courses.

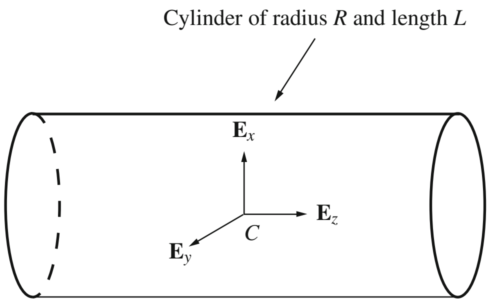

Fig. 8.11 *A circular cylinder of mass m, radius R, and length L.*

above expression for $\mathbf{H}$ simplifies considerably:

$$\begin{bmatrix} \mathbf{H}\cdot\mathbf{e}_x \\ \mathbf{H}\cdot\mathbf{e}_y \\ \mathbf{H}\cdot\mathbf{e}_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega}\cdot\mathbf{e}_x \\ \boldsymbol{\omega}\cdot\mathbf{e}_y \\ \boldsymbol{\omega}\cdot\mathbf{e}_z \end{bmatrix}.$$

If possible, one chooses $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to be the sets of principal axes.

8.8.3 A Circular Cylinder

As an example, we consider a rigid homogeneous circular cylinder of mass m, radius R, and length L shown in Figure 8.11. For this body, we choose $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ to be the principal axes of the body: $I_{xy} = I_{xz} = I_{yz} = 0$. The center of mass C of the cylinder is at its geometric center.

Evaluating the volume integrals discussed previously, one obtains

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}mR^2 + \frac{1}{12}mL^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 + \frac{1}{12}mL^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix}.$$

Notice that by setting $R = 0$ one can use these results to determine the inertias of a slender rod. Similarly, by setting $L = 0$, the inertias for a thin circular disk can be obtained.

8.8.4 The Parallel-Axis Theorem

In many problems, it is convenient to consider the inertia matrix relative to a point A that is not the center of mass. The relevant inertias $I_{A_{xx}}, \ldots, I_{A_{zz}}$ can be determined from the components of the inertia matrix for the body relative to its center of mass using the parallel-axis theorem (see, for instance, Appendix B of Meriam and Kraige [48]). When A is a point of the rigid body, the inertias $I_{A_{xx}}, \ldots, I_{A_{zz}}$ can then be used to determine a convenient expression for $\mathbf{H}_A$. Here, we find it more convenient to use the relationship

$$\mathbf{H}_A = \int_{\mathscr{R}} (\mathbf{x} - \mathbf{x}_A) \times \mathbf{v} \rho dv = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_A) \times \mathbf{G}.$$

Here, $\mathbf{x}_A$ is the position vector of the point A. This equation holds even if A is not a material point of the rigid body. Moreover, it subsumes the parallel-axis theorem. A specific example is discussed in Section 9.4 of Chapter 9.

8.9 Some Comments on Integrals and Derivatives

In several of the developments in Sections 8.4, 8.7, and 8.8 some identities were used that are beyond the scope of an undergraduate engineering dynamics course. Here, we present some further details on these identities for the interested reader.

A difficulty with evaluating the expressions for $\bar{\mathbf{x}}$ and $\mathbf{H}$ is that the region of integration $\mathscr{R}$ depends on time. We have similar issues in the next chapter when evaluating the derivatives of certain integrals. We now record some results pertaining to these integrals when the motion of the body is rigid: $\mathbf{x} = \chi_R(\mathbf{X}, t)$. Proofs of these results can be found in the literature on continuum mechanics.[18]

First, we have the local conservation of mass result

$$\rho_0(\mathbf{X}) = \rho(\mathbf{x} = \chi_R(\mathbf{X}, t), t).$$

In words, this implies that the mass density at a material point X of the rigid body is the same in its reference and present configurations.

The second result is a change of variables theorem for any sufficiently smooth function f:

$$\int_{\mathscr{R}} f(\mathbf{x}, t) dv = \int_{\mathscr{R}_0} f(\chi_R(\mathbf{X}, t), t) dV.$$

We used this result and mass conservation to establish expressions for the inertias in Section 8.8.

The last result is a version of Reynolds' transport theorem for a sufficiently smooth function g:

$$\frac{d}{dt} \int_{\mathscr{R}} g(\mathbf{x}, t) \rho dv = \int_{\mathscr{R}} \frac{d}{dt} (g(\mathbf{x}, t)) \rho dv.$$

[18] These results follow from the local forms of mass conservation, changes of variables theorem, and Reynolds' transport theorem in continuum mechanics because a rigid body's motion is isochoric (see, e.g., Casey [13], Gurtin [33], and Truesdell and Toupin [80]).

We used this result in Section 8.4 when we took the time derivative outside the integral to establish the result that $\mathbf{G} = m\bar{\mathbf{v}}$.

In Section 8.4 we used the result that the center of mass of a rigid body behaves as if it were a material point. This result is accepted without comment by most texts. The first proof of it, to our knowledge, is by Casey [13]. Our outline here is far longer than his proof. The reason for this is that he uses a compact tensor notation. We first recall from Section 8.1 that for any two material points X and Y of a rigid body,

$$\begin{bmatrix} (\mathbf{x}-\mathbf{y})\cdot\mathbf{E}_x \\ (\mathbf{x}-\mathbf{y})\cdot\mathbf{E}_y \\ (\mathbf{x}-\mathbf{y})\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} (\mathbf{X}-\mathbf{Y})\cdot\mathbf{E}_x \\ (\mathbf{X}-\mathbf{Y})\cdot\mathbf{E}_y \\ (\mathbf{X}-\mathbf{Y})\cdot\mathbf{E}_z \end{bmatrix}.$$

One now substitutes these results into the right-hand side of the identity

$$\bar{\mathbf{x}}-\mathbf{y} = \frac{1}{m}\int_{\mathscr{R}} \mathbf{x}\rho dv - \mathbf{y} = \frac{1}{m}\int_{\mathscr{R}} \mathbf{x}\rho dv - \frac{\int_{\mathscr{R}} \rho dv}{m}\mathbf{y} = \frac{1}{m}\int_{\mathscr{R}} (\mathbf{x}-\mathbf{y})\rho dv.$$

Next, one uses the change of variables result recorded above and the definition of $\bar{\mathbf{X}}$ to conclude that

$$\begin{bmatrix} (\bar{\mathbf{x}}-\mathbf{y})\cdot\mathbf{E}_x \\ (\bar{\mathbf{x}}-\mathbf{y})\cdot\mathbf{E}_y \\ (\bar{\mathbf{x}}-\mathbf{y})\cdot\mathbf{E}_z \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} (\bar{\mathbf{X}}-\mathbf{Y})\cdot\mathbf{E}_x \\ (\bar{\mathbf{X}}-\mathbf{Y})\cdot\mathbf{E}_y \\ (\bar{\mathbf{X}}-\mathbf{Y})\cdot\mathbf{E}_z \end{bmatrix}.$$

This result implies that the center of mass C of the rigid body behaves as if it were a material point of the body. For deformable bodies, this is not true.[19]

8.10 Summary

The first part of this chapter was devoted to examining the kinematical relationships among the position vectors, velocity vectors, and acceleration vectors of two material points X_1 and X_2 of a rigid body. Then expressions for the linear momentum $\mathbf{G}$ and angular momenta $\mathbf{H}$, $\mathbf{H}_O$, and $\mathbf{H}_A$ were presented. Expressions for the angular momenta were simplified using the inertia tensor. For pedagogical reasons, we found it convenient to present many of the results for arbitrary rotations and then simplify them for the case of a fixed-axis of rotation.

Denoting the position vectors of the material points X_1 and X_2 by $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, it was shown that

$$\mathbf{v}_1 - \mathbf{v}_2 = \boldsymbol{\omega}\times(\mathbf{x}_1-\mathbf{x}_2),$$
$$\mathbf{a}_1 - \mathbf{a}_2 = \boldsymbol{\alpha}\times(\mathbf{x}_1-\mathbf{x}_2) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{x}_1-\mathbf{x}_2)),$$

[19] For example, take a flexible ruler. Initially, suppose that it is straight. One can approximately locate its center of mass; suppose that it is at the geometric center. Now, bend the ruler into a circle. The center of mass no longer coincides with the same material point of the ruler.

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ is the angular acceleration vector of the rigid body. To facilitate working with several problems, a corotational basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ was also introduced. This basis rotates with the body. We also showed that $\dot{\mathbf{e}}_x = \boldsymbol{\omega} \times \mathbf{e}_x$, $\dot{\mathbf{e}}_y = \boldsymbol{\omega} \times \mathbf{e}_y$, and $\dot{\mathbf{e}}_z = \boldsymbol{\omega} \times \mathbf{e}_z$. In Section 8.4, the center of mass C of the rigid body was introduced. For a rigid body, C behaves as a material point of the rigid body. In addition, the linear momentum of the rigid body is

$$\mathbf{G} = m\bar{\mathbf{v}},$$

where $\bar{\mathbf{v}}$ is the velocity vector of the center of mass and m is the mass of the rigid body. The angular momentum of a rigid body relative to its center of mass has the representation

$$\begin{aligned}\mathbf{H} = &\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)\mathbf{e}_x + \left(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)\mathbf{e}_y \\ &+ \left(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right)\mathbf{e}_z.\end{aligned}$$

The inertias $I_{xx}, \ldots, I_{zz}$ are constants associated with the rigid body. They depend on the mass and geometry of the rigid body and the choice of the basis vectors $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$. If possible these vectors are chosen to be principal axes of the body, in which case $I_{xy} = I_{yz} = I_{xz} = 0$. It was also shown that

$$\mathbf{H}_O = \mathbf{H} + \bar{\mathbf{x}} \times \mathbf{G}, \qquad \mathbf{H}_A = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_A) \times \mathbf{G}.$$

Most of the results in this chapter were specialized to the case of a fixed-axis of rotation. This axis was chosen to be $\mathbf{E}_z$, and the angle of rotation of the rigid body was denoted by θ. All of the aforementioned kinematical results simplify for this case. For instance,

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z, \qquad \boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z,$$

and

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_y = \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x, \qquad \mathbf{e}_z = \mathbf{E}_z.$$

The most substantial simplification occurs in the expression for $\mathbf{H}$:

$$\mathbf{H} = I_{xz}\dot{\theta}\mathbf{e}_x + I_{yz}\dot{\theta}\mathbf{e}_y + I_{zz}\dot{\theta}\mathbf{E}_z.$$

The kinematical results presented in the chapter were used to examine the kinematics of mechanisms and rolling and sliding rigid bodies. In the mechanism problem discussed in Section 8.3, it was shown how to determine the angular velocities and angular accelerations of two of the linkages in a four-bar mechanism as functions of the angular velocity and acceleration of a third linkage. For rolling and sliding rigid bodies, the important conditions $\mathbf{v}_P = \mathbf{0}$ and $\mathbf{v}_P \cdot \mathbf{n} = 0$ were discussed.

8.11 Exercises

The following short exercises are intended to assist you in reviewing the present chapter. In the exercises, we restrict attention to the fixed-axis of rotation case: $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$.

8.1. Show that

$$\dot{\mathbf{e}}_x = \dot{\theta}\mathbf{e}_y, \qquad \dot{\mathbf{e}}_y = -\dot{\theta}\mathbf{e}_x, \qquad \dot{\mathbf{e}}_z = \mathbf{0}.$$

8.2. The position vector of a material point of a rigid body relative to a fixed point O of the rigid body is

$$\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z.$$

By differentiating this result, show that the expressions for $\mathbf{v}$ and $\mathbf{a}$ are identical to those you would have obtained had you used the formulae

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}, \qquad \mathbf{a} = \boldsymbol{\alpha} \times \mathbf{x} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}).$$

8.3. A particle of mass m is free to move in a groove that is machined on a rigid body. The center of mass C of the rigid body is fixed. If the position vector of the particle relative to C is

$$\mathbf{r} - \bar{\mathbf{x}} = 5\mathbf{e}_x + y\mathbf{e}_y,$$

calculate $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$. Here, y is not a constant.

8.4. How do the results of Exercise 8.3 change if the center of mass C has a motion $\bar{\mathbf{x}} = 10t\mathbf{E}_x + t^2\mathbf{E}_y$?

8.5. For the mechanism discussed in Section 8.3, determine the angular velocities and accelerations of the links DC and BC at the instant where

$$\theta_1 = \frac{\pi}{2} \text{ radians}, \qquad \dot{\theta}_1 = 1.0 \text{ RPM}, \qquad \ddot{\theta}_1 = 0.1 \text{ radians/sec}^2.$$

The dimensions of the linkages are

$$L_1 = 10 \text{ meters}, \qquad L_2 = 5 \text{ meters}, \qquad L_3 = 10 \text{ meters}.$$

8.6. For the rolling disk discussed in Section 8.6, determine the velocity and acceleration vectors of the material point X whose position vector relative to the center of mass is

$$\mathbf{x} - \bar{\mathbf{x}} = R\mathbf{e}_x.$$

At an instant during each revolution of the disk, why does X have a velocity vector $\mathbf{0}$?

8.7. For the sliding disk discussed in Section 8.6, determine the velocity and acceleration vectors of the material point X whose position vector relative to the center of mass is

$$\mathbf{x} - \bar{\mathbf{x}} = R\mathbf{e}_x.$$

When is it possible for X to have the same acceleration and velocity as the center of mass?

8.8. For the rolling conditions on a circular disk discussed in Section 8.6, many people, in haste, write $\dot{x} = R\dot{\theta}$. Show that this result does not imply that $\mathbf{v}_P \cdot \mathbf{E}_x = 0$. What does it imply?

8.9. For the rolling circular disk discussed in Section 8.6, derive expressions for $\mathbf{H}$ and $\mathbf{H}_P$.

8.10. Calculate the time derivative of

$$\mathbf{H} = I_{xz}\dot{\theta}\mathbf{e}_x + I_{yz}\dot{\theta}\mathbf{e}_y + I_{zz}\dot{\theta}\mathbf{E}_z.$$

The answer is displayed in Section 9.1.3 of Chapter 9.

Chapter 9
Kinetics of a Rigid Body

TOPICS

We start by discussing Euler's laws for a rigid body. These laws are known as the balances of linear and angular momenta. An alternative form of these laws is also presented that is useful for solving many classes of problems. We then discuss the kinetic energy of a rigid body and establish the Koenig decomposition. This decomposition, combined with the balance laws, can be used to prove a work-energy theorem for a rigid body. As illustrations of the theory we consider four classes of problems: purely translational motion of a rigid body, rigid bodies that are free to rotate about one of their fixed material points, rolling and sliding bodies, and an imbalanced rotor problem. These applications are far from exhaustive, but we feel they are the chief representatives of problems for an undergraduate engineering dynamics course.

9.1 Balance Laws for a Rigid Body

Euler's laws for a rigid body can be viewed as extensions to Newton's second law for a particle. There are two laws (postulates): the balance of linear momentum and the balance of angular momentum.[1]

[1] For a single particle or system of particles, the balance of angular momentum is not an independent postulate: as shown in Chapter 7, it follows from the balance of linear momentum. You should also notice that we do not attempt, as many texts do, to derive the balances of linear and angular momenta from the balance of linear momentum for each of the material points of a rigid body. Such a derivation entails placing restrictions on the nature of the internal forces acting on the particles. Here, we follow the approach in continuum mechanics and postulate two independent balance laws (see Essay V of Truesdell [79] for further discussions on this matter).

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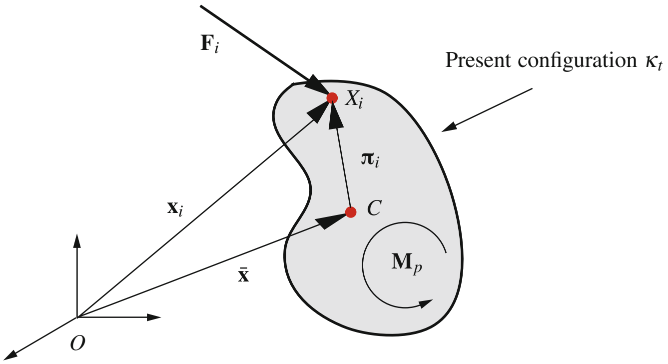

Fig. 9.1 *A force* $\mathbf{F}_i$ *and a moment* $\mathbf{M}_p$ *acting on a rigid body.*

9.1.1 Resultant Forces and Moments

Before discussing the balance laws, we dispense with some preliminaries. The resultant force $\mathbf{F}$ acting on the rigid body is the sum of all the forces acting on the rigid body. Similarly, the resultant moment relative to a fixed point O, $\mathbf{M}_O$, is the resultant external moment relative to O of all of the moments acting on the rigid body. We also denote the resultant moment relative to the center of mass C by $\mathbf{M}$. These moments may be decomposed into two additive parts: the moment due to the individual external forces acting on the rigid body and applied external moments that are not due to external forces.

As an example, consider a system of forces and moments acting on a rigid body. Here, a set of K forces $\mathbf{F}_i$ $(i = 1, \ldots, K)$ act on the rigid body. The force $\mathbf{F}_i$ acts at the material point X_i, which has a position vector $\mathbf{x}_i$. In addition, a moment $\mathbf{M}_p$, which is not due to the moment of an applied force, acts on the rigid body (see Figure 9.1). For this system of applied forces and moments, the resultants are

$$\mathbf{F} = \sum_{i=1}^{K} \mathbf{F}_i,$$

$$\mathbf{M}_O = \mathbf{M}_p + \sum_{i=1}^{K} \mathbf{x}_i \times \mathbf{F}_i,$$

$$\mathbf{M} = \mathbf{M}_p + \sum_{i=1}^{K} (\mathbf{x}_i - \bar{\mathbf{x}}) \times \mathbf{F}_i.$$

Notice how $\mathbf{M}_p$ features in these expressions.

9.1.2 Euler's Laws

The balance laws for a rigid body are often known as Euler's laws.[2] The first of these laws is the balance of linear momentum:

$$\mathbf{F} = \dot{\mathbf{G}} = m\dot{\bar{\mathbf{v}}}.$$

The second law is the balance of angular momentum for a rigid body:

$$\mathbf{M}_O = \dot{\mathbf{H}}_O.$$

The pair of balance laws represent six scalar equations.

In many cases it is convenient to give an alternative description of the balance of angular momentum. To do this we start with the identity

$$\mathbf{H}_O = \mathbf{H} + \bar{\mathbf{x}} \times \mathbf{G}.$$

Differentiating and invoking the balance of linear momentum, we find that

$$\begin{aligned} \dot{\mathbf{H}}_O &= \dot{\mathbf{H}} + \bar{\mathbf{v}} \times \mathbf{G} + \bar{\mathbf{x}} \times \dot{\mathbf{G}} \\ &= \dot{\mathbf{H}} + \bar{\mathbf{x}} \times \mathbf{F}. \end{aligned}$$

Hence, invoking the balance of angular momentum,

$$\mathbf{M}_O = \dot{\mathbf{H}}_O = \dot{\mathbf{H}} + \bar{\mathbf{x}} \times \mathbf{F}.$$

However, the resultant moment relative to a fixed point O, $\mathbf{M}_O$, and the resultant moment relative to the center of mass C, $\mathbf{M}$, are related by[3]

$$\mathbf{M}_O = \mathbf{M} + \bar{\mathbf{x}} \times \mathbf{F}.$$

It follows that

$$\mathbf{M} = \dot{\mathbf{H}},$$

which is known as the balance of angular momentum relative to the center of mass C. This form of the balance law is used in many problems where the rigid body has no fixed point O.

We now recall the developments of Section 8.8 of Chapter 8, where we introduced the inertia tensor:

$$\begin{aligned} \mathbf{H} = &\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)\mathbf{e}_x + \left(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)\mathbf{e}_y \\ &+ \left(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right)\mathbf{e}_z. \end{aligned}$$

[2] See Truesdell [79]. For a rigid body, they may be seen on pages 224–225 of another seminal paper [26] by Euler which was published in 1776.

[3] This may be seen from our previous discussion of a system of forces and moments acting on a rigid body.

Here, $I_{xx}, \ldots, I_{zz}$ are the components of the inertia tensor of the rigid body relative to the center of mass and the angular velocity vector of the rigid body is $\boldsymbol{\omega} = \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z$. In the balance laws, the derivative of $\mathbf{H}$ is required. This is obtained by differentiating the expression for $\mathbf{H}$ above. The resulting expression is conveniently written in the form

$$\dot{\mathbf{H}} = \overset{o}{\mathbf{H}} + \boldsymbol{\omega} \times \mathbf{H},$$

where $\overset{o}{\mathbf{H}}$ is the corotational rate of $\mathbf{H}$. This is the time derivative of $\mathbf{H}$ that is obtained while keeping $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ fixed:

$$\begin{aligned}\overset{o}{\mathbf{H}} = &\left(I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z\right)\mathbf{e}_x + \left(I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z\right)\mathbf{e}_y \\ &+ \left(I_{xz}\dot{\omega}_x + I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z\right)\mathbf{e}_z.\end{aligned}$$

Clearly, the resulting balance of angular momentum $\mathbf{M} = \dot{\mathbf{H}}$ gives rise to a complex set of equations.

9.1.3 *The Fixed-Axis of Rotation Case*

Simplifying the aforementioned results to the fixed-axis of rotation case:

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_y = \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x, \qquad \mathbf{e}_z = \mathbf{E}_z,$$

$$\dot{\mathbf{e}}_x = \dot{\theta}\mathbf{e}_y, \qquad \dot{\mathbf{e}}_y = -\dot{\theta}\mathbf{e}_x, \qquad \boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z = \omega\mathbf{E}_z.$$

Furthermore,

$$\begin{aligned}\mathbf{H} &= I_{xz}\omega\mathbf{e}_x + I_{yz}\omega\mathbf{e}_y + I_{zz}\omega\mathbf{E}_z, \\ \dot{\mathbf{H}} &= \left(I_{xz}\dot{\omega} - I_{yz}\omega^2\right)\mathbf{e}_x + \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{E}_z.\end{aligned}$$

The balance laws for the fixed-axis of rotation case can be written as

$$\begin{aligned}\mathbf{F} &= m\dot{\bar{\mathbf{v}}}, \\ \mathbf{M} &= \left(I_{xz}\dot{\omega} - I_{yz}\omega^2\right)\mathbf{e}_x + \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{E}_z.\end{aligned}$$

The first three of these equations, $\mathbf{F} = m\dot{\bar{\mathbf{v}}}$, give the motion of the center of mass and any reaction forces acting on the body. The fourth and fifth equations ($\mathbf{M}\cdot\mathbf{e}_x = \dot{\mathbf{H}}\cdot\mathbf{e}_x$ and $\mathbf{M}\cdot\mathbf{e}_y = \dot{\mathbf{H}}\cdot\mathbf{e}_y$) give the reaction moment $\mathbf{M}_c$ which ensures that the rotation of the rigid body is only about the $\mathbf{E}_z$-axis. Last, but not least, the sixth equation ($\mathbf{M}\cdot\mathbf{E}_z = \dot{\mathbf{H}}\cdot\mathbf{E}_z$) gives a differential equation for $\theta(t)$.

9.1.4 The Four Steps

In solving problems, we follow the four steps used earlier for particles. There are some modifications:

1. Pick an origin, a coordinate system, and a corotational basis, and then establish expressions for $\mathbf{H}$ (or $\mathbf{H}_O$), $\bar{\mathbf{x}}$, $\bar{\mathbf{v}}$, and $\bar{\mathbf{a}}$.
2. Draw a free-body diagram showing the external forces $\mathbf{F}_i$ and moments $\mathbf{M}_p$.
3. Write out the six equations $\mathbf{F} = m\bar{\mathbf{a}}$ and $\mathbf{M} = \dot{\mathbf{H}}$ (or $\mathbf{M}_O = \dot{\mathbf{H}}_O$).
4. Perform the analysis.

These steps will guide you through most problems. We emphasize once more that the free-body diagram is only used as an aid to checking one's solution.

It is a common beginner's mistake to use the balance of angular momentum about a point, say A, that is neither the center of mass C nor fixed $(\mathbf{v}_A \neq \mathbf{0})$. If one does this then it is important to note that $\mathbf{M}_A \neq \dot{\mathbf{H}}_A$, rather $\mathbf{M}_A = \dot{\mathbf{H}}_A + \mathbf{v}_A \times \mathbf{G}$.

9.2 Work-Energy Theorem and Energy Conservation

Here, we first show the Koenig decomposition for the kinetic energy of a rigid body:

$$T = \frac{1}{2}m\bar{\mathbf{v}}\cdot\bar{\mathbf{v}} + \frac{1}{2}\mathbf{H}\cdot\boldsymbol{\omega}.$$

This is then followed by a development of the work-energy theorem for a rigid body:

$$\frac{dT}{dt} = \mathbf{F}\cdot\bar{\mathbf{v}} + \mathbf{M}\cdot\boldsymbol{\omega} = \sum_{i=1}^{K}\mathbf{F}_i\cdot\mathbf{v}_i + \mathbf{M}_p\cdot\boldsymbol{\omega}.$$

As in particles and systems of particles, this theorem can be used to establish conservation of the total energy of a rigid body during a motion.

9.2.1 Koenig's Decomposition

We begin with Koenig's[4] decomposition of the kinetic energy of a rigid body.[5] By definition, the kinetic energy T of a rigid body is

$$T = \frac{1}{2}\int_{\mathscr{R}}\mathbf{v}\cdot\mathbf{v}\rho\, dv.$$

[4] Johann Samuel Koenig (1712–1757) was a German mathematican and philosopher. He was also a contemporary of Euler.

[5] Our proof of Koenig's decomposition follows Casey [13, 16].

We next recall that the velocity vector of any material point X of the rigid body has the representation

$$\mathbf{v} = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \boldsymbol{\pi},$$

where the relative position vector $\boldsymbol{\pi}$ and the angular velocity vector $\boldsymbol{\omega}$ are

$$\boldsymbol{\pi} = \mathbf{x} - \bar{\mathbf{x}}, \qquad \boldsymbol{\omega} = \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z.$$

Substituting for $\mathbf{v}$ in the expression for T and expanding, we find that

$$T = \frac{1}{2}\int_{\mathscr{R}} \left(\bar{\mathbf{v}} \cdot \bar{\mathbf{v}} + 2\bar{\mathbf{v}} \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) + (\boldsymbol{\omega} \times \boldsymbol{\pi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) \right) \rho dv.$$

However,

$$\frac{1}{2}\int_{\mathscr{R}} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} \rho dv = \frac{\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}}{2}\int_{\mathscr{R}} \rho dv = \frac{1}{2} m \bar{\mathbf{v}} \cdot \bar{\mathbf{v}},$$

$$\int_{\mathscr{R}} \bar{\mathbf{v}} \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) \rho dv = \bar{\mathbf{v}} \cdot \left(\boldsymbol{\omega} \times \int_{\mathscr{R}} \boldsymbol{\pi} \rho dv \right) = \bar{\mathbf{v}} \cdot (\boldsymbol{\omega} \times \mathbf{0}) = 0.$$

Consequently,

$$T = \frac{1}{2} m \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} + \frac{1}{2}\int_{\mathscr{R}} (\boldsymbol{\omega} \times \boldsymbol{\pi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) \rho dv.$$

We can simplify the second expression of the right-hand side of this equation using a vector identity:

$$(\boldsymbol{\omega} \times \boldsymbol{\pi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\pi}) = ((\boldsymbol{\pi} \cdot \boldsymbol{\pi})\boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega})\boldsymbol{\pi}) \cdot \boldsymbol{\omega}.$$

Substituting this result into the expression for T and rearranging, we find that

$$T = \frac{1}{2} m \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} + \frac{1}{2}\left(\int_{\mathscr{R}} ((\boldsymbol{\pi} \cdot \boldsymbol{\pi})\boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega})\boldsymbol{\pi}) \rho dv \right) \cdot \boldsymbol{\omega}.$$

Recall from Section 8.8 of Chapter 8 that the angular momentum relative to the center of mass C of the rigid body is

$$\mathbf{H} = \int_{\mathscr{R}} \boldsymbol{\pi} \times (\boldsymbol{\omega} \times \boldsymbol{\pi}) \rho dv = \int_{\mathscr{R}} \left((\boldsymbol{\pi} \cdot \boldsymbol{\pi})\boldsymbol{\omega} - (\boldsymbol{\pi} \cdot \boldsymbol{\omega})\boldsymbol{\pi} \right) \rho dv.$$

Hence, we obtain the Koenig decomposition:

$$T = \frac{1}{2} m \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} + \frac{1}{2}\mathbf{H} \cdot \boldsymbol{\omega}.$$

In words, the kinetic energy of a rigid body can be decomposed into the sum of the rotational kinetic energy and the translational kinetic energy of the center of mass.

9.2.2 The Work-Energy Theorem

To establish a work-energy theorem for a rigid body, we start by differentiating T:

$$\dot{T} = \frac{1}{2}m\dot{\bar{\mathbf{v}}}\cdot\bar{\mathbf{v}} + \frac{1}{2}m\bar{\mathbf{v}}\cdot\dot{\bar{\mathbf{v}}} + \frac{1}{2}\dot{\mathbf{H}}\cdot\boldsymbol{\omega} + \frac{1}{2}\mathbf{H}\cdot\dot{\boldsymbol{\omega}}.$$

To proceed further, we need to show that $\dot{\mathbf{H}}\cdot\boldsymbol{\omega} = \mathbf{H}\cdot\dot{\boldsymbol{\omega}}$. In the preceding pages all of these terms except for one are recorded. The only missing term is

$$\begin{aligned}\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} &= \frac{d}{dt}\left(\omega_x\mathbf{e}_x + \omega_y\mathbf{e}_y + \omega_z\mathbf{e}_z\right)\\ &= \dot{\omega}_x\mathbf{e}_x + \dot{\omega}_y\mathbf{e}_y + \dot{\omega}_z\mathbf{e}_z + \omega_x\dot{\mathbf{e}}_x + \omega_y\dot{\mathbf{e}}_y + \omega_z\dot{\mathbf{e}}_z\\ &= \dot{\omega}_x\mathbf{e}_x + \dot{\omega}_y\mathbf{e}_y + \dot{\omega}_z\mathbf{e}_z + \boldsymbol{\omega}\times\left(\omega_x\mathbf{e}_x + \omega_y\mathbf{e}_y + \omega_z\mathbf{e}_z\right)\\ &= \dot{\omega}_x\mathbf{e}_x + \dot{\omega}_y\mathbf{e}_y + \dot{\omega}_z\mathbf{e}_z.\end{aligned}$$

Another direct calculation using this expression for $\boldsymbol{\alpha}$ shows that

$$\begin{aligned}\mathbf{H}\cdot\dot{\boldsymbol{\omega}} = &\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)\dot{\omega}_x + \left(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)\dot{\omega}_y\\ &+ \left(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right)\dot{\omega}_z.\end{aligned}$$

Comparing this to the corresponding expression for $\dot{\mathbf{H}}\cdot\boldsymbol{\omega}$, we find that they are equal. Consequently,

$$\dot{T} = \frac{1}{2}m\dot{\bar{\mathbf{v}}}\cdot\bar{\mathbf{v}} + \frac{1}{2}m\bar{\mathbf{v}}\cdot\dot{\bar{\mathbf{v}}} + \frac{1}{2}\dot{\mathbf{H}}\cdot\boldsymbol{\omega} + \frac{1}{2}\dot{\mathbf{H}}\cdot\boldsymbol{\omega}.$$

This result implies that

$$\dot{T} = m\dot{\bar{\mathbf{v}}}\cdot\bar{\mathbf{v}} + \dot{\mathbf{H}}\cdot\boldsymbol{\omega}.$$

Invoking the balance of linear momentum and the balance of angular momentum, we obtain the work-energy theorem:

$$\dot{T} = \mathbf{F}\cdot\bar{\mathbf{v}} + \mathbf{M}\cdot\boldsymbol{\omega}.$$

You should notice how this is a natural extension of the work-energy theorem for a single particle.

9.2.3 An Alternative Form of the Work-Energy Theorem

In applications, it is convenient to use an equivalent form of the work-energy theorem. This can be obtained by substituting for the moments and forces discussed at the beginning of Section 9.1:

$$\mathbf{F} = \sum_{i=1}^{K} \mathbf{F}_i, \qquad \mathbf{M} = \mathbf{M}_p + \sum_{i=1}^{K} (\mathbf{x}_i - \bar{\mathbf{x}}) \times \mathbf{F}_i.$$

Hence, the mechanical power of the resultant forces and moments can be written as

$$\mathbf{F} \cdot \bar{\mathbf{v}} + \mathbf{M} \cdot \boldsymbol{\omega} = \left(\sum_{i=1}^{K} \mathbf{F}_i \right) \cdot \bar{\mathbf{v}} + \mathbf{M}_p \cdot \boldsymbol{\omega} + \left(\sum_{i=1}^{K} (\mathbf{x}_i - \bar{\mathbf{x}}) \times \mathbf{F}_i \right) \cdot \boldsymbol{\omega}.$$

With some minor manipulations involving the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ and noting that $\mathbf{v}_i = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{x}_i - \bar{\mathbf{x}})$, we find that

$$\mathbf{F} \cdot \bar{\mathbf{v}} + \mathbf{M} \cdot \boldsymbol{\omega} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}.$$

In conclusion, we have an alternative form of the work-energy theorem that proves to be far easier to use in applications:

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}.$$

We present examples shortly involving this theorem to illustrate how it is used to establish conservation of energy results.

It is crucial to note from the work-energy theorem that the mechanical powers of a force $\mathbf{P}$ acting at a material point X whose position vector is $\mathbf{x}$ and a moment $\mathbf{L}$ are

Mechanical power of a force $\mathbf{P}$ acting at $\mathbf{x}$: $\mathbf{P} \cdot \dot{\mathbf{x}}$,
Mechanical power of a moment $\mathbf{L}$: $\mathbf{L} \cdot \boldsymbol{\omega}$.

These expressions can be easily used to determine whether a force or moment is workless and thus does not contribute to the change of kinetic energy of the rigid body during a motion.

9.3 Purely Translational Motion of a Rigid Body

A rigid body performing a purely translational motion is arguably the simplest class of problems associated with these bodies. For these problems, the angular velocity $\boldsymbol{\omega}$ and acceleration $\boldsymbol{\alpha}$ vectors are $\mathbf{0}$. The velocity and acceleration of any material point of the rigid body are none other than those for the center of mass C. Recalling the expression for the angular momentum $\mathbf{H}$,

$$\begin{aligned}\mathbf{H} = &\left(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\right)\mathbf{e}_x + \left(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\right)\mathbf{e}_y \\ &+ \left(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z\right)\mathbf{e}_z,\end{aligned}$$

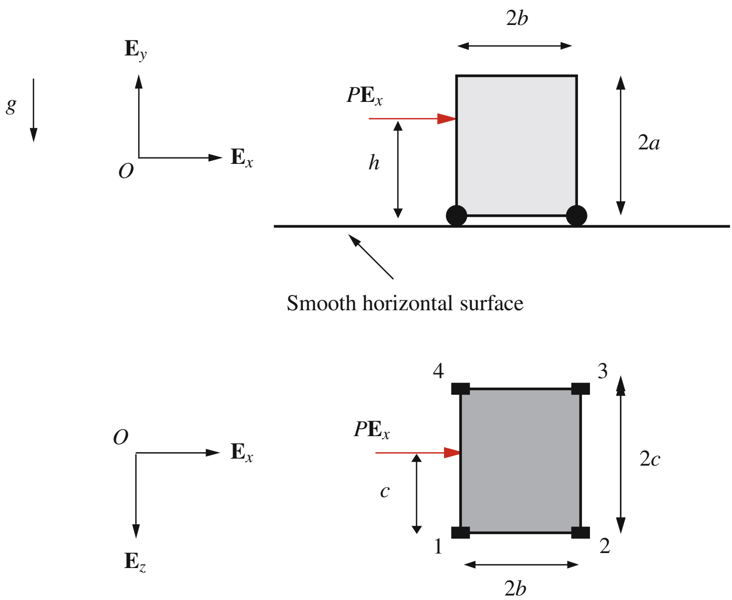

Fig. 9.2 *A side view and a plan view of a cart moving on a smooth horizontal track.*

one also finds that, for these problems, $\mathbf{H}$ and $\dot{\mathbf{H}}$ are $\mathbf{0}$.

For purely translational problems, the balance laws are simply

$$\mathbf{F} = m\dot{\bar{\mathbf{v}}}, \qquad \mathbf{M} = \mathbf{0}.$$

These give six equations that are used to solve for the constraint forces and moments and the motion of the center of mass of the rigid body. In addition, the work-energy theorem is simply

$$\dot{T} = \mathbf{F} \cdot \bar{\mathbf{v}}.$$

9.3.1 The Overturning Cart

We now consider an example. As shown in Figure 9.2, a cart of mass m, height $2a$, width $2b$, and depth $2c$ is being driven by a force $\mathbf{P} = P\mathbf{E}_x$. This force is applied to a point on one of its sides. The cart is free to move on a smooth horizontal track. We wish to determine the restrictions on P such that the cart will not topple. In addition, we prove that the total energy of the cart is conserved.

You should notice how the solution of this problem follows the four steps we discussed earlier.

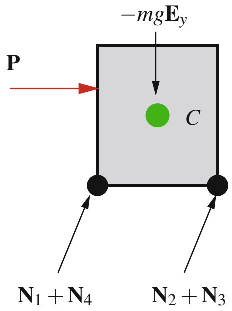

Fig. 9.3 *Free-body diagram of the cart.*

9.3.1.1 Kinematics

We start with the kinematics and choose a Cartesian coordinate system to describe $\bar{\mathbf{x}}$:

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y_0\mathbf{E}_y + z_0\mathbf{E}_z,$$

where y_0 and z_0 are constant because we only consider the case where all four wheels of the cart remain in contact with the track. Differentiating this expression gives the velocity and acceleration vectors of the center of mass. For the present problem, one does not need to establish an expression for $\mathbf{H}$. Furthermore, we find no need to explicitly mention the existence of a corotational basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ because we can choose $\mathbf{e}_x = \mathbf{E}_x$, $\mathbf{e}_y = \mathbf{E}_y$, and $\mathbf{e}_z = \mathbf{E}_z$.

9.3.1.2 Forces and Moments

We next consider the free-body diagram (shown in Figure 9.3). Notice that there are four reaction forces, one on each of the four wheels: $\mathbf{N}_i$ acts on the wheel numbered i in Figure 9.2, where $i = 1$, 2, 3, or 4. These forces have components in the $\mathbf{E}_y$ and $\mathbf{E}_z$ directions.[6]

The resultant force acting on the system is

$$\mathbf{F} = P\mathbf{E}_x - mg\mathbf{E}_y + \left(N_{1_y} + N_{2_y} + N_{3_y} + N_{4_y}\right)\mathbf{E}_y + \left(N_{1_z} + N_{2_z} + N_{3_z} + N_{4_z}\right)\mathbf{E}_z.$$

[6] The components of the normal forces in the $\mathbf{E}_z$ direction ensure that the cart will not start to rotate about the $\mathbf{E}_y$-axis.

The resultant moment can be calculated by taking the moments of these forces about the center of mass:

$$\begin{aligned}\mathbf{M} = &((h-a)\mathbf{E}_y - b\mathbf{E}_x) \times P\mathbf{E}_x + \mathbf{0} \times -mg\mathbf{E}_y \\ &+ (-a\mathbf{E}_y - b\mathbf{E}_x + c\mathbf{E}_z) \times \left(N_{1_y}\mathbf{E}_y + N_{1_z}\mathbf{E}_z\right) \\ &+ (-a\mathbf{E}_y + b\mathbf{E}_x + c\mathbf{E}_z) \times \left(N_{2_y}\mathbf{E}_y + N_{2_z}\mathbf{E}_z\right) \\ &+ (-a\mathbf{E}_y + b\mathbf{E}_x - c\mathbf{E}_z) \times \left(N_{3_y}\mathbf{E}_y + N_{3_z}\mathbf{E}_z\right) \\ &+ (-a\mathbf{E}_y - b\mathbf{E}_x - c\mathbf{E}_z) \times \left(N_{4_y}\mathbf{E}_y + N_{4_z}\mathbf{E}_z\right).\end{aligned}$$

With some work, the expression for $\mathbf{M}$ can be simplified:

$$\begin{aligned}\mathbf{M} = &(a-h)P\mathbf{E}_z + c(-N_{1_y} - N_{2_y} + N_{3_y} + N_{4_y})\mathbf{E}_x \\ &-a(N_{1_z} + N_{2_z} + N_{3_z} + N_{4_z})\mathbf{E}_x + b(N_{1_z} - N_{2_z} - N_{3_z} + N_{4_z})\mathbf{E}_y \\ &+b(-N_{1_y} + N_{2_y} + N_{3_y} - N_{4_y})\mathbf{E}_z.\end{aligned}$$

9.3.1.3 Balance Laws

We now invoke the balance laws and take their components with respect to $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ to obtain the six equations

$$\begin{aligned}P &= m\ddot{x}, \\ -mg + N_{1_y} + N_{2_y} + N_{3_y} + N_{4_y} &= 0, \\ N_{1_z} + N_{2_z} + N_{3_z} + N_{4_z} &= 0, \\ -c(N_{1_y} + N_{2_y} - N_{3_y} - N_{4_y}) &= a(N_{1_z} + N_{2_z} + N_{3_z} + N_{4_z}), \\ N_{1_z} - N_{2_z} - N_{3_z} + N_{4_z} &= 0, \\ (h-a)P &= b(-N_{1_y} + N_{2_y} + N_{3_y} - N_{4_y}).\end{aligned}$$

There are nine unknowns: $x(t)$, N_{1_y}, N_{2_y}, N_{3_y}, N_{4_y}, N_{1_z}, N_{2_z}, N_{3_z}, and N_{4_z}. It follows that the system of equations is indeterminate and additional assumptions may be required to solve the problem.

9.3.1.4 Analysis

First, let's determine the motion of the system. From the six equations given above, we see that

$$\ddot{x} = \frac{P}{m}.$$

Subject to the initial conditions at $t = 0$ that $\bar{\mathbf{x}}(0) = y_0\mathbf{E}_y + z_0\mathbf{E}_z$ and $\bar{\mathbf{v}}(0) = v_0\mathbf{E}_x$, we can integrate this equation to find that the motion of the center of mass is

$$\bar{\mathbf{x}} = \left(v_0 t + \frac{Pt^2}{2m}\right)\mathbf{E}_x + y_0\mathbf{E}_y + z_0\mathbf{E}_z.$$

In words, the cart's center of mass is accelerated in the direction of the applied force $P\mathbf{E}_x$, a result that shouldn't surprise you.

Next, we seek to determine the toppling force. From the six equations above, we obtain five equations for the eight unknown reaction forces. As mentioned earlier, this is an indeterminate system of equations. In the sequel, we can ignore the horizontal forces $N_{1_z}, N_{2_z}, N_{3_z}$, and N_{4_z}. To proceed to solve for the other unknown forces ($N_{1_y}, N_{2_y}, N_{3_y}$, and N_{4_y}) we need to make two additional assumptions.[7] These assumptions relate the forces on the individual wheels:

$$N_{1_y} = N_{4_y}, \qquad N_{2_y} = N_{3_y}.$$

That is, the vertical forces on the rear wheels are identical and the vertical forces on the front wheels are identical. Solving for the reaction forces we find that

$$N_{1_y} = N_{4_y} = \frac{1}{4}\left(mg - \frac{P}{b}(h-a)\right),$$
$$N_{2_y} = N_{3_y} = \frac{1}{4}\left(mg + \frac{P}{b}(h-a)\right).$$

To determine the allowable range of P, we set the $\mathbf{E}_y$ components of the reaction forces to zero. Consequently, the front wheels will lose contact, provided that $N_{2y} = N_{3y} < 0$:

$$P < -\frac{mgb}{h-a}.$$

Similarly, the rear wheels lose contact when

$$P > \frac{mgb}{h-a}.$$

Observe that if $h = a$, then the reaction forces never vanish and overturning in this case never occurs. For a given cart and $h \neq a$, one can now easily determine the allowable range of P. We leave this as an exercise. In the course of this exercise, notice that if a cart is tall (i.e., $a \gg b$) then the toppling force P is smaller than if the cart were stout (i.e., $a \ll b$).

[7] Of course, the correct approach here would be to model each of the four wheels attached to the cart as individual rigid bodies. Then, instead of modeling this system as a single rigid body, one has a system of five rigid bodies. Unfortunately, one has a similar indeterminacy in this model also. In vehicle system dynamics, an area that is primarily concerned with modeling automobiles using interconnected rigid bodies, this issue is usually not seen because one incorporates suspension models (see Gillespie [30]).

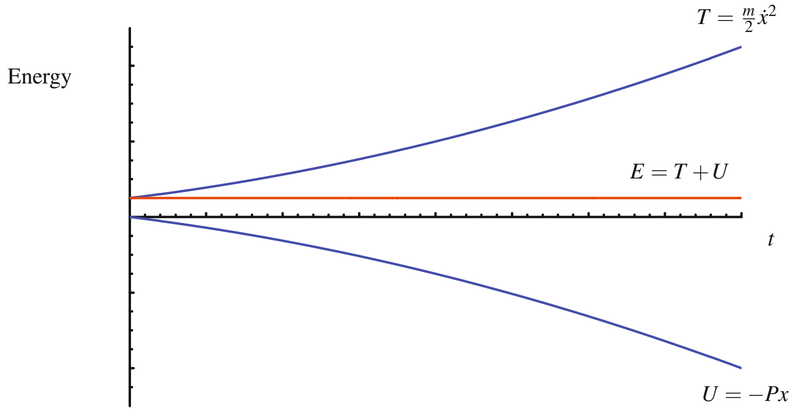

Fig. 9.4 *Conservation of the total energy E of the cart.*

We now address energy conservation for this problem. Starting from the work-energy theorem, we find that

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \bar{\mathbf{v}} = (-mg + N_{1_y} + N_{2_y} + N_{3_y} + N_{4_y})\mathbf{E}_y \cdot \bar{\mathbf{v}} + (N_{1_z} + N_{2_z} + N_{3_z} + N_{4_z})\mathbf{E}_z \cdot \bar{\mathbf{v}} + P\mathbf{E}_x \cdot \bar{\mathbf{v}}.$$

However, the normal forces and gravity have no mechanical power in this problem and P is constant:

$$\dot{T} = P\mathbf{E}_x \cdot \bar{\mathbf{v}} = \frac{d}{dt}\left(P\mathbf{E}_x \cdot \bar{\mathbf{x}}\right).$$

Hence, the total energy E of this system is conserved:

$$\frac{d}{dt}\left(E = T - P\mathbf{E}_x \cdot \bar{\mathbf{x}} = \frac{1}{2}m\dot{x}^2 - Px\right) = 0.$$

For this problem, the rotational kinetic energy of the cart is zero: $T = 0.5m\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}$. The energy conservation of the cart is illustrated in Figure 9.4 for a specific example where $\dot{x}(0) > 0$ and $x(0) = 0$.

9.4 A Rigid Body with a Fixed Point

In the class of problems considered in this section, a rigid body is attached at one of its material points by a pin-joint to a fixed point (cf. Figure 9.5). Here, we take this fixed point to be the origin O of our coordinate system. At the pin-joint there is a reaction force $\mathbf{R}$ and a reaction moment $\mathbf{M}_c$. These ensure that the point of

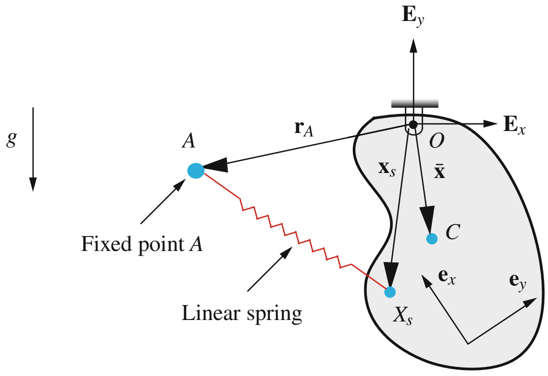

Fig. 9.5 *A representative example of a rigid body rotating about a fixed point O.*

attachment remains fixed and the axis of rotation of the rigid body remains fixed, respectively.

The example shown in Figure 9.5 is representative. We assume that a gravitational force $-mg\mathbf{E}_y$ acts on the body, in addition to a spring force:

$$\mathbf{F}_s = -K(\|\mathbf{x}_s - \mathbf{r}_A\| - L_0)\frac{\mathbf{x}_s - \mathbf{r}_A}{\|\mathbf{x}_s - \mathbf{r}_A\|}.$$

One end of the spring is attached to the material point X_s of the rigid body whose position vector is $\mathbf{x}_s$. The other end is attached to a fixed point A whose position vector is $\mathbf{r}_A$.

Restricting attention to planar motions, we seek to determine the differential equation governing the motion of this rigid body, the reaction forces, and the reaction moments. We also prove that the total energy of this rigid body is conserved.

9.4.1 Kinematics

Because the motion is constrained to be planar, $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z = \omega\mathbf{E}_z$ and $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z$. The corotational basis is defined in the usual manner:

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_y = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y, \qquad \mathbf{e}_z = \mathbf{E}_z.$$

We also recall the relations

$$\dot{\mathbf{e}}_x = \dot{\theta}\mathbf{e}_y, \qquad \dot{\mathbf{e}}_y = -\dot{\theta}\mathbf{e}_x, \qquad \dot{\mathbf{e}}_z = \mathbf{0}.$$

Taking the fixed point O as the origin, we denote the position vector of the center of mass of the body by

$$\bar{\mathbf{x}} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z.$$

Here, x, y, and z are constants. Differentiating this expression, we find that

$$\bar{\mathbf{v}} = \dot{\theta}(x\mathbf{e}_y - y\mathbf{e}_x), \qquad \bar{\mathbf{a}} = \ddot{\theta}(x\mathbf{e}_y - y\mathbf{e}_x) - \dot{\theta}^2(x\mathbf{e}_x + y\mathbf{e}_y).$$

We also define the position vectors of the spring's attachment points:

$$\mathbf{x}_s = x_s\mathbf{e}_x + y_s\mathbf{e}_y + z_s\mathbf{e}_z, \qquad \mathbf{r}_A = X_A\mathbf{E}_x + Y_A\mathbf{E}_y + Z_A\mathbf{E}_z,$$

where all of the displayed coordinates are constant. To keep the development clear, we avoid explicit use of these representations for $\mathbf{x}_s$ and $\mathbf{r}_A$.

Next, we address the angular momenta of the rigid body. Because the axis of rotation is fixed, we recall from Section 9.1 that

$$\begin{aligned}\mathbf{H} &= I_{xz}\omega\mathbf{e}_x + I_{yz}\omega\mathbf{e}_y + I_{zz}\omega\mathbf{E}_z,\\ \dot{\mathbf{H}} &= \left(I_{xz}\dot{\omega} - I_{yz}\omega^2\right)\mathbf{e}_x + \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{E}_z.\end{aligned}$$

It is convenient in this problem to determine $\mathbf{H}_O$. To this end, we recall that

$$\mathbf{H}_O = \mathbf{H} + \bar{\mathbf{x}} \times \mathbf{G}.$$

Substituting for the problem of interest, we obtain

$$\mathbf{H}_O = I_{xz}\omega\mathbf{e}_x + I_{yz}\omega\mathbf{e}_y + I_{zz}\omega\mathbf{E}_z + (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \times m\dot{\theta}(x\mathbf{e}_y - y\mathbf{e}_x).$$

With some rearranging, we find that

$$\mathbf{H}_O = (I_{xz} - mxz)\,\omega\mathbf{e}_x + (I_{yz} - myz)\,\omega\mathbf{e}_y + \left(I_{zz} + m\left(x^2 + y^2\right)\right)\omega\mathbf{E}_z.$$

These results are identical to those that would have been obtained had one used the parallel-axis theorem to determine the moment of inertia tensor of the body about point O.

For future purposes, we also determine the kinetic energy T of the rigid body. We start from the Koenig decomposition and substitute for the various kinematical quantities to find that

$$\begin{aligned}T &= \frac{1}{2}m\bar{\mathbf{v}}\cdot\bar{\mathbf{v}} + \frac{1}{2}\mathbf{H}\cdot\boldsymbol{\omega}\\ &= \frac{1}{2}\left(I_{zz} + m\left(x^2 + y^2\right)\right)\dot{\theta}^2.\end{aligned}$$

Notice that $T = \frac{1}{2}\mathbf{H}_O\cdot\boldsymbol{\omega}$ for this problem.

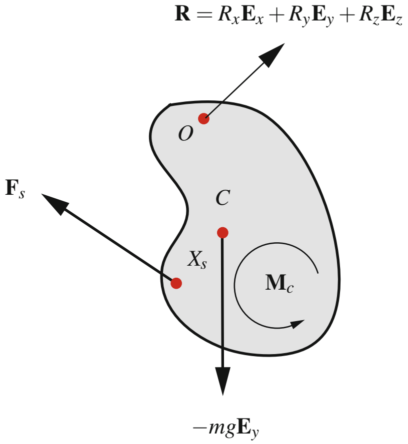

Fig. 9.6 *Free-body diagram.*

9.4.2 Forces and Moments

The free-body diagram for the body is shown in Figure 9.6. Notice that there is a reaction force $\mathbf{R}$ at O:

$$\mathbf{R} = R_x\mathbf{E}_x + R_y\mathbf{E}_y + R_z\mathbf{E}_z.$$

In addition, there is a reaction moment $\mathbf{M}_c$ acting on the body:

$$\mathbf{M}_c = M_{c_x}\mathbf{e}_x + M_{c_y}\mathbf{e}_y.$$

The moment $\mathbf{M}_c$ ensures that the rotation of the body is constrained to being about the $\mathbf{E}_z$ axis. This is normally not mentioned in engineering dynamics texts. We shortly show why it is needed. The resultant force and moment on the body are

$$\mathbf{F} = -mg\mathbf{E}_y + \mathbf{F}_s + \mathbf{R}, \qquad \mathbf{M}_O = \mathbf{M}_c - \bar{\mathbf{x}} \times mg\mathbf{E}_y + \mathbf{x}_s \times \mathbf{F}_s.$$

9.4.3 Balance Laws

We now turn to the balances of linear and angular momenta for the rigid body. Due to the unknown force $\mathbf{R}$, it is easiest to use the balance of angular momentum about O. The balance laws are

$$\mathbf{F} = m\bar{\mathbf{a}}, \qquad \mathbf{M}_O = \dot{\mathbf{H}}_O.$$

Substituting for the resultant forces and moments, we find that

$$\mathbf{F}_s - mg\mathbf{E}_y + \mathbf{R} = m\bar{\mathbf{a}}, \qquad \mathbf{M}_c - \bar{\mathbf{x}} \times mg\mathbf{E}_y + \mathbf{x}_s \times \mathbf{F}_s = \dot{\mathbf{H}}_O.$$

We have refrained from substituting for the inertias and accelerations here. These six equations may be solved for the five unknown reactions, $\mathbf{R}$ and $\mathbf{M}_c$, and they also provide a differential equation for $\theta(t)$.

9.4.4 Analysis

We first determine the unknown forces and moments. From the balance of linear momentum, we obtain three equations for the three unknown components of $\mathbf{R}$:

$$\begin{aligned}\mathbf{R} &= mg\mathbf{E}_y - \mathbf{F}_s + m\bar{\mathbf{a}} \\ &= mg\mathbf{E}_y + K\left(\|\mathbf{x}_s - \mathbf{r}_A\| - L_0\right)\frac{\mathbf{x}_s - \mathbf{r}_A}{\|\mathbf{x}_s - \mathbf{r}_A\|} + m\left(\ddot{\theta}(x\mathbf{e}_y - y\mathbf{e}_x) - \dot{\theta}^2(x\mathbf{e}_x + y\mathbf{e}_y)\right).\end{aligned}$$

Next, we find, from the balance of angular momentum, that

$$\begin{aligned}M_{c_x} = \mathbf{M}_c \cdot \mathbf{e}_x &= \left(-\mathbf{x}_s \times \mathbf{F}_s + \bar{\mathbf{x}} \times mg\mathbf{E}_y + \dot{\mathbf{H}}_O\right) \cdot \mathbf{e}_x, \\ M_{c_y} = \mathbf{M}_c \cdot \mathbf{e}_y &= \left(-\mathbf{x}_s \times \mathbf{F}_s + \bar{\mathbf{x}} \times mg\mathbf{E}_y + \dot{\mathbf{H}}_O\right) \cdot \mathbf{e}_y.\end{aligned}$$

The expression for the spring force $\mathbf{F}_s$ and rate of change of angular momentum $\dot{\mathbf{H}}_O$ can be substituted into these expressions if desired. Notice that if the reaction moment were omitted, then the $\mathbf{e}_x$ and $\mathbf{e}_y$ components of the balance of angular momentum would not be satisfied. As a result the body could not move as one had assumed when setting up the kinematics of the problem.[8]

The motion of the rigid body can be found from the sole remaining equation. This equation is the $\mathbf{e}_z = \mathbf{E}_z$ component of the balance of angular momentum:

$$(\mathbf{x}_s \times \mathbf{F}_s - \bar{\mathbf{x}} \times mg\mathbf{E}_y) \cdot \mathbf{E}_z = \dot{\mathbf{H}}_O \cdot \mathbf{E}_z.$$

Substituting for the forces and momentum, we find that

$$\begin{aligned}\left(I_{zz} + m\left(x^2 + y^2\right)\right)\ddot{\theta} &= -mg\left(x\cos(\theta) - y\sin(\theta)\right) \\ &\quad + K\left(\|\mathbf{x}_s - \mathbf{r}_A\| - L_0\right)\frac{(\mathbf{x}_s \times \mathbf{r}_A) \cdot \mathbf{E}_z}{\|\mathbf{x}_s - \mathbf{r}_A\|}.\end{aligned}$$

Given the initial conditions $\theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = \omega_0$, the solution of this equation determines the motion of the body.[9]

For this type of problem, one could also take the balance of angular momentum relative to the center of mass C. In this case, the reaction force $\mathbf{R}$ would make its

[8] The error in neglecting $\mathbf{M}_c$ is equivalent to ignoring the normal force acting on a particle that is assumed to move on a surface.

[9] This is a nonlinear differential equation. As mentioned for a related example in Appendix A, its analytical solution can be found and expressed in terms of Jacobi's elliptic functions. The interested reader is referred to Lawden [43] or Whittaker [81] for details on these functions and how they are used in the solution of dynamics problems.

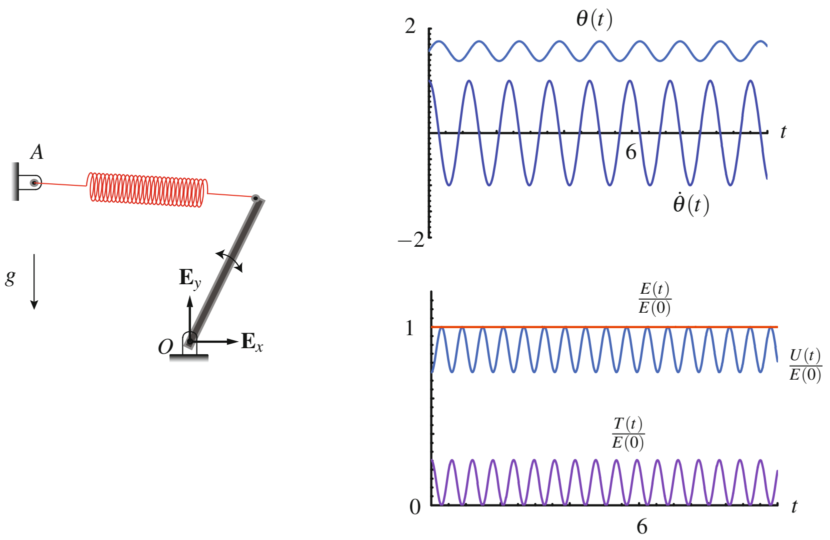

Fig. 9.7 *The oscillatory motion of a rod that is attached to a fixed point A by a linear spring. Plots of $\theta(t)$, $\dot{\theta}(t)$, $T(t)$, $U(t)$, and $E(t)$ are also shown.*

presence known in all six equations. By taking the balance of angular momentum relative to the point O, this reaction force does not enter three of the six equations and, consequently, makes the system of six equations easier to solve.

Let us now prove why the total energy E of the system is conserved. We start with the (alternative form of the) work-energy theorem and substitute for the forces and moments:

$$\dot{T} = \mathbf{R}\cdot\mathbf{v}_O - mg\mathbf{E}_y\cdot\bar{\mathbf{v}} + \mathbf{M}_c\cdot\boldsymbol{\omega} + \mathbf{F}_s\cdot\mathbf{v}_s.$$

An expression for the kinetic energy T for this problem was recorded earlier. Now, because O is fixed, $\mathbf{v}_O = \mathbf{0}$, so $\mathbf{R}$ has no mechanical power. Similarly, $\mathbf{M}_c$ is perpendicular to $\boldsymbol{\omega}$, so it too has no power. The spring and gravitational forces are conservative:

$$-mg\mathbf{E}_y\cdot\bar{\mathbf{v}} = -\frac{d}{dt}\left(mg\mathbf{E}_y\cdot\bar{\mathbf{x}}\right), \qquad \mathbf{F}_s\cdot\mathbf{v}_s = -\frac{d}{dt}\left(\frac{K}{2}\left(\|\mathbf{x}_s - \mathbf{r}_A\| - L_0\right)^2\right).$$

Combining these results, we find that the total energy E of the rigid body is conserved:

$$\frac{d}{dt}\left(E = T + mg\mathbf{E}_y\cdot\bar{\mathbf{x}} + \frac{K}{2}\left(\|\mathbf{x}_s - \mathbf{r}_A\| - L_0\right)^2\right) = 0.$$

One uses this equation in a similar manner as with particles. For instance, given the initial conditions $\theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = \omega_0$, one can then use the conservation of E to determine $\dot{\theta}$ at a later instant of the motion when θ is given.

To illustrate the results of this section, we consider a rod of length L and mass m which is pin-jointed at O. The other end of the rod is attached to a fixed point A by a spring of stiffness K and unstretched length $L_0 = L$. Here, $\mathbf{r}_A = -L\mathbf{E}_x + L\mathbf{E}_y$. The differential equation governing $\theta(t)$ can be easily derived using the results presented previously.[10] We assume that the rod is initially vertical and then set it in motion with a small initial velocity. Provided the spring is sufficiently stiff, the rod will oscillate about the vertical. An example of this situation is shown in Figure 9.7. In one of the plots shown in this figure, energy conservation during this motion is illustrated:

$$T = \frac{mL^2}{3}\dot{\theta}^2, \qquad U = \frac{mgL}{2}\sin(\theta) + \frac{K}{2}\left(\|\mathbf{x}_s - \mathbf{r}_A\| - L\right)^2, \qquad E = T + U.$$

The other plot shows the oscillatory behaviors of $\theta(t)$ and $\dot{\theta}(t)$. The periodic nature of $\theta(t)$ is one of the reasons why systems similar to the one shown in Figure 9.7 are often used in timing devices such as metronomes.

9.5 Rolling and Sliding Rigid Bodies

We now return to the rolling and sliding rigid bodies considered in Sections 8.5 and 8.6 of Chapter 8. We start by considering a rigid body $\mathscr{B}$ that is in motion atop a fixed surface $\mathscr{S}$ (see Figure 9.8). At the instantaneous point of contact $P = X_P(t)$, the outward unit normal to the surface is $\mathbf{n}$. The velocity vector of the material point $X_P(t)$ that is instantaneously in contact with the surface is denoted by $\mathbf{v}_P$. Earlier, we saw that if the rigid body is rolling on the surface, then one has the rolling condition:

$$\mathbf{v}_P = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}}) = \mathbf{0}.$$

If the rigid body is sliding on the surface, then one has the sliding condition:

$$\mathbf{v}_P \cdot \mathbf{n} = \bar{\mathbf{v}} \cdot \mathbf{n} + (\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}})) \cdot \mathbf{n} = 0.$$

We now turn to the forces that enforce these constraints.

9.5.1 Friction

The force at the instantaneous point of contact P depends on the natures of the outer surface of the rigid body and the fixed surface. If their contact is smooth, then the reaction force at P is

$$\mathbf{F}_P = N\mathbf{n}.$$

[10] For this example, $I_{zz} = mL^2/12$, $x = L/2$, $y = 0$, $\mathbf{x}_s = L\mathbf{e}_x$, and $L_0 = L$. For the simulations shown in Figure 9.7, the following parameter values were used: $m = 1$ kilogram, $L = 10$ meters, and $K = 100$ Newtons per meter.

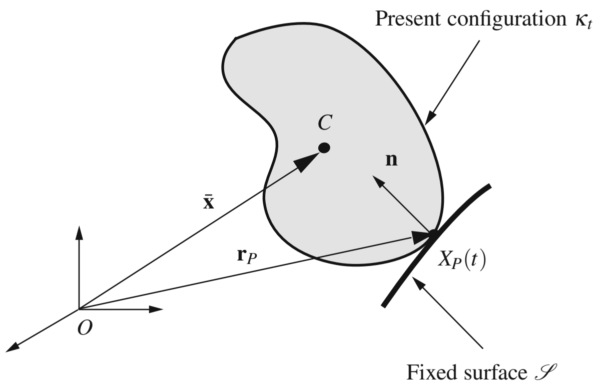

Fig. 9.8 *The geometry of contact.*

On the other hand, if the surface is rough with coefficients of friction μ_s and μ_d, then this force is either of the static type ($\mathbf{v}_P = \mathbf{0}$),

$$\mathbf{F}_P = F_{p_x}\mathbf{E}_x + F_{p_y}\mathbf{E}_y + F_{p_z}\mathbf{E}_z \quad \text{where } \|\mathbf{F}_P - (\mathbf{F}_P \cdot \mathbf{n})\mathbf{n}\| \leq \mu_s |\mathbf{F}_P \cdot \mathbf{n}|,$$

or, if there is relative motion ($\mathbf{v}_P \neq \mathbf{0}$),

$$\mathbf{F}_P = N\mathbf{n} - \mu_d \|N\mathbf{n}\| \frac{\mathbf{v}_P}{\|\mathbf{v}_P\|}.$$

Clearly, if the contact is rough, then the rigid body can either roll or slip depending primarily on the amount of static friction available.

9.5.2 Energy Considerations

If the contact is smooth, then it should be clear that the force $\mathbf{F}_P$ is workless. Similarly, if the rigid body is rolling, then the mechanical power of $\mathbf{F}_P$ is $\mathbf{F}_P \cdot \mathbf{v}_P = 0$. It is only when the rigid body is sliding and the contact is rough that $\mathbf{F}_P$ does work:

$$\mathbf{F}_P \cdot \mathbf{v}_P = N\mathbf{n} \cdot \mathbf{v}_P - \mu_d \|N\mathbf{n}\| \frac{\mathbf{v}_P}{\|\mathbf{v}_P\|} \cdot \mathbf{v}_P = -\mu_d \|N\mathbf{n}\| \|\mathbf{v}_P\| < 0.$$

Notice that the power of the force in this case is negative, so it will decrease the kinetic energy T.

The previous results imply that if the only other forces acting on a rolling rigid body are conservative (such as gravitational and spring forces), then the total energy of the rigid body will be conserved. A similar comment applies to a sliding rigid body when the contact is smooth. As a result, in most solved problems in this area,

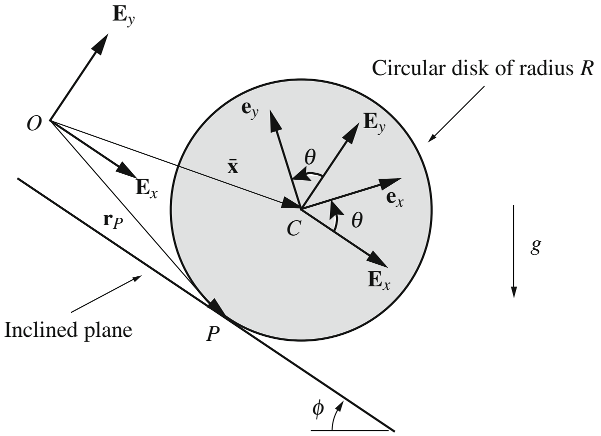

Fig. 9.9 *A rigid body in motion on an inclined plane.*

such as sliding disks and tops, and rolling disks and spheres, this energy conservation is present.[11]

9.6 Examples of Rolling and Sliding Rigid Bodies

Let us now turn to a specific example, which is shown in Figure 9.9. We consider a rigid body of mass m whose outer surface is circular with radius R. We assume that the rigid body moves on an inclined plane under the influence of gravity. The center of mass C of the rigid body is assumed to be located at the geometric center of the circle of radius R. The contact is assumed to be rough with coefficients of friction μ_s and μ_d. It is assumed that $\mu_s \geq \mu_d$.

9.6.1 General Considerations

We assume that the center of mass of the rigid body is given an initial velocity $v_0\mathbf{E}_x$, where $v_0 > 0$, at time $t = 0$, and we seek to determine the motion of the rigid body for subsequent times.

[11] There are few completely solved problems in this area (see, for examples and references, Borisov and Mamaev [9, 10], Hermans [34], Neimark and Fufaev [51], O'Reilly [55], Papastavridis [58], Routh [64, 65], and Zenkov et al. [82]).

9.6.1.1 Kinematics

We assume that the motion is such that the axis of rotation is fixed: $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ and $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z$. The corotational basis is defined in the usual manner:

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \quad \mathbf{e}_y = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y, \quad \mathbf{e}_z = \mathbf{E}_z.$$

We also recall the relations

$$\dot{\mathbf{e}}_x = \dot{\theta}\mathbf{e}_y, \qquad \dot{\mathbf{e}}_y = -\dot{\theta}\mathbf{e}_x, \qquad \dot{\mathbf{e}}_z = \mathbf{0}.$$

Taking the origin as shown in Figure 9.9, we denote the position vector of the center of mass of the body by

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y\mathbf{E}_y + z_0\mathbf{E}_z.$$

Here, z_0 is a constant. We shortly show that y is also a constant. Differentiating the expression for $\bar{\mathbf{x}}$ we find that

$$\bar{\mathbf{v}} = \dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y, \qquad \bar{\mathbf{a}} = \ddot{x}\mathbf{E}_x + \ddot{y}\mathbf{E}_y.$$

Next, we address the angular momentum of the rigid body. We assume that $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are principal axes of the rigid body in its present configuration. Hence,

$$\mathbf{H} = I_{zz}\dot{\theta}\mathbf{E}_z, \qquad \dot{\mathbf{H}} = I_{zz}\ddot{\theta}\mathbf{E}_z.$$

If the contact condition is such that sliding occurs, then, as the normal $\mathbf{n} = \mathbf{E}_y$,

$$\mathbf{v}_P \cdot \mathbf{E}_y = \bar{\mathbf{v}} \cdot \mathbf{E}_y + (\boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}})) \cdot \mathbf{E}_y = 0.$$

For the present problem, $\mathbf{r}_P - \bar{\mathbf{x}} = -R\mathbf{E}_y$, so the sliding condition implies that $\dot{y} = 0$, as expected. For sliding, one has

$$\mathbf{v}_P = \left(\dot{x} + R\dot{\theta}\right)\mathbf{E}_x = v_P\mathbf{E}_x.$$

The velocity v_P is often referred to as the slip velocity. On the other hand, if rolling occurs, then

$$\mathbf{v}_P = \bar{\mathbf{v}} + \boldsymbol{\omega} \times (\mathbf{r}_P - \bar{\mathbf{x}}) = \mathbf{0}.$$

This implies for the present problem that

$$\bar{\mathbf{v}} = \dot{x}\mathbf{E}_x = -R\dot{\theta}\mathbf{E}_x.$$

We discussed these results previously in Section 8.6 of Chapter 8.

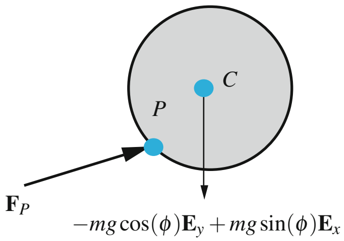

Fig. 9.10 *Free-body diagram.*

9.6.1.2 Forces and Moments

One of the tasks remaining is to describe **F** and **M**. The free-body diagram for the system is shown in Figure 9.10. In sum,

$$\begin{aligned}\mathbf{F} &= -mg\cos(\phi)\mathbf{E}_y + mg\sin(\phi)\mathbf{E}_x + \mathbf{F}_P,\\ \mathbf{M} &= -R\mathbf{E}_y \times \mathbf{F}_P.\end{aligned}$$

If the body is rolling, then

$$\mathbf{F}_P = F_{P_x}\mathbf{E}_x + F_{P_y}\mathbf{E}_y + F_{P_z}\mathbf{E}_z,$$

whereas if the body is sliding, then

$$\mathbf{F}_P = N\mathbf{E}_y - \mu_d \|N\mathbf{E}_y\| \frac{\mathbf{v}_P}{\|\mathbf{v}_P\|} = N\mathbf{E}_y - \mu_d |N| \frac{\dot{x} + R\dot{\theta}}{|\dot{x} + R\dot{\theta}|}\mathbf{E}_x.$$

9.6.1.3 Balance Laws

We are now in a position to examine the balance laws:

$$\begin{aligned}\mathbf{F}_P - mg\cos(\phi)\mathbf{E}_y + mg\sin(\phi)\mathbf{E}_x &= m\ddot{x}\mathbf{E}_x,\\ -R\mathbf{E}_y \times \mathbf{F}_P &= I_{zz}\ddot{\theta}\mathbf{E}_z.\end{aligned}$$

These equations are used to determine the force at P and the motion of the rigid body in what follows.

9.6.2 The Rolling Case

We first find, from the balance of linear momentum, that

$$\mathbf{F}_P = mg\cos(\phi)\mathbf{E}_y - mg\sin(\phi)\mathbf{E}_x + m\ddot{x}\mathbf{E}_x.$$

Substituting this result into the balance of angular momentum and using the condition $\ddot{x} = -R\ddot{\theta}$, one obtains a differential equation for $x(t)$:

$$\left(\frac{I_{zz} + mR^2}{R}\right)\ddot{x} = mgR\sin(\phi).$$

This equation can be easily solved to determine $x(t)$:

$$x(t) = \left(\frac{mgR^2\sin(\phi)}{2(I_{zz} + mR^2)}\right)t^2 + v_0 t.$$

In writing this solution, we assumed that $x(t = 0) = 0$. We have also tacitly assumed, in order to satisfy the rolling condition, that $\dot{\theta}(t = 0) = -v_0/R$. We can also find the friction and normal forces as functions of time by substituting for $x(t)$ in the expression for $\mathbf{F}_P$ given above:

$$\mathbf{F}_P = mg\cos(\phi)\mathbf{E}_y - \left(\frac{mg\sin(\phi)I_{zz}}{I_{zz} + mR^2}\right)\mathbf{E}_x.$$

This is the complete solution to the rolling case.[12]

9.6.3 The Static Friction Criterion and Rolling

To determine whether there is a transition to sliding, we need to check the magnitude of the friction force for the rolling rigid body. Here, we use the standard static friction criterion:

$$\left\|F_{P_x}\mathbf{E}_x\right\| \leq \mu_s \left\|F_{P_y}\mathbf{E}_y\right\|.$$

Substituting for $\mathbf{F}_P$, we find that

$$\left\|\left(-\frac{mg\sin(\phi)I_{zz}}{I_{zz} + mR^2}\right)\mathbf{E}_x\right\| \leq \mu_s \left\|mg\cos(\phi)\mathbf{E}_y\right\|.$$

This criterion can be simplified to

$$\beta\tan(\phi) \leq \mu_s,$$

where the parameter β is

$$\beta = \frac{I_{zz}}{I_{zz} + mR^2}.$$

[12] It is a good exercise to show that the total energy of this rolling rigid body is conserved. If this rigid body slips, then you should also be able to show that, because $\mathbf{F}_P \cdot \mathbf{v}_P < 0$, the total energy will decrease with time.

If the incline is sufficiently steep, or the inertia is distributed in a certain manner, then $\beta\tan(\phi) > \mu_s$. Now suppose a body is placed at rest on an incline where $\beta\tan(\phi) \leq \mu_s$. The body will immediately start to roll down the incline and, because there is sufficient static friction, it will not start to slip.

9.6.4 The Sliding Case

For a sliding rigid body, the balance laws yield

$$
\begin{aligned}
N\mathbf{E}_y - \mu_d\,|N|\,\frac{\dot{x}+R\dot{\theta}}{\left|\dot{x}+R\dot{\theta}\right|}\mathbf{E}_x - mg\cos(\phi)\mathbf{E}_y + mg\sin(\phi)\mathbf{E}_x &= m\ddot{x}\mathbf{E}_x, \\
-R\mathbf{E}_y \times \left(N\mathbf{E}_y - \mu_d|N|\frac{\dot{x}+R\dot{\theta}}{\left|\dot{x}+R\dot{\theta}\right|}\mathbf{E}_x\right) &= I_{zz}\ddot{\theta}\mathbf{E}_z.
\end{aligned}
$$

From these equations we obtain differential equations for x and θ and an equation for the normal force $N\mathbf{E}_y$.

First, for the normal force, we find that $N\mathbf{E}_y = mg\cos(\phi)\mathbf{E}_y$. Next, the differential equations are

$$
\begin{aligned}
-\mu_d mg\cos(\phi)\mathrm{sgn}(v_P) + mg\sin(\phi) &= m\ddot{x}, \\
-\mu_d mgR\cos(\phi)\mathrm{sgn}(v_P) &= I_{zz}\ddot{\theta},
\end{aligned}
$$

where

$$
\mathrm{sgn}(v_P) = \frac{\dot{x}+R\dot{\theta}}{\left|\dot{x}+R\dot{\theta}\right|},
$$

and $v_P = \dot{x}+R\dot{\theta}$ is the slip velocity. Solving the differential equations subject to the initial conditions $x(t=0)=0$, $\theta\,(t=0)=0$, $\dot{x}\,(t=0)=v_0>0$, and $\dot{\theta}\,(t=0)=\omega_0$, we find that

$$
\begin{aligned}
x(t) &= \frac{g\cos(\phi)}{2}\left(\tan(\phi) - \mu_d\mathrm{sgn}(v_P)\right)t^2 + v_0 t, \\
\theta(t) &= -\mu_d\left(\frac{mgR\cos(\phi)}{2I_{zz}}\right)\mathrm{sgn}(v_P)t^2 + \omega_0 t.
\end{aligned}
$$

Thus, we have determined the motion of the center of mass and the rotation of the rigid body.

For problems featuring sliding rigid bodies it is often illuminating to determine $v_P(t)$. To arrive at this expression, we first determine a differential equation for $\dot{v}_P$

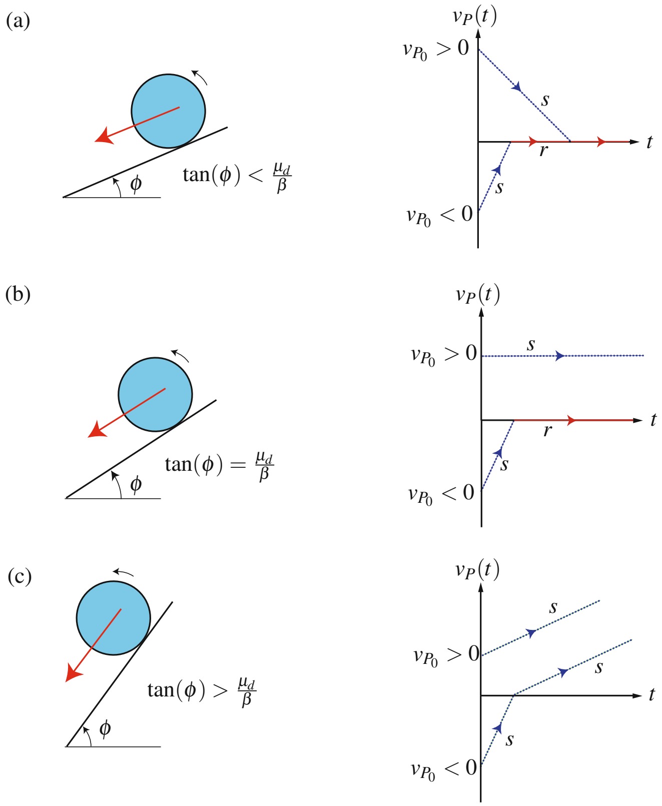

Fig. 9.11 *The evolution of the slip velocity $v_P = \dot{x} + R\dot{\theta}$: (a) $\beta\tan(\phi) < \mu_d = \mu_s$ and there is sufficient friction to enable a transition from sliding to rolling, (b) $\beta\tan(\phi) = \mu_d = \mu_s$, and (c) $\beta\tan(\phi) > \mu_d = \mu_s$.* The label s denotes sliding, the label r denotes rolling, v_{P_0} denotes $v_P(t=0)$, and $\beta = I_{zz}/\left(I_{zz} + mR^2\right)$.

from the earlier differential equations for $\ddot{\theta}$ and $\ddot{x}$:

$$\begin{aligned}\dot{v}_P &= \ddot{x} + R\ddot{\theta} \\ &= \frac{g\cos(\phi)}{\beta}\left(\beta\tan(\phi) - \mu_d \operatorname{sgn}(v_P)\right).\end{aligned}$$

As shown in Figure 9.11, if $v_P < 0$, then the equation for $\dot{v}_P$ implies that $v_P(t)$ will increase linearly in time. On the other hand, when $v_P > 0$, then the behavior of $v_P(t)$ depends on the ratio $\beta\tan(\phi)/\mu_d$ where $\beta = I_{zz}/\left(I_{zz} + mR^2\right)$. Here, we assume that $\mu_d = \mu_s$. From our discussion in Section 9.6.3, when $\beta\tan(\phi) < \mu_s$ there is sufficient friction to ensure that rolling can be sustained. Referring to Figure 9.11(a), we observe that $v_P \rightarrow 0$ and when $v_P = 0$, rolling starts and persists indefinitely.[13] On

[13] This type of transition occurs in bowling balls as they approach the pins. To aid this phenomenon, the bowling lanes are usually waxed to increase the friction between the ball and the lane as the for-

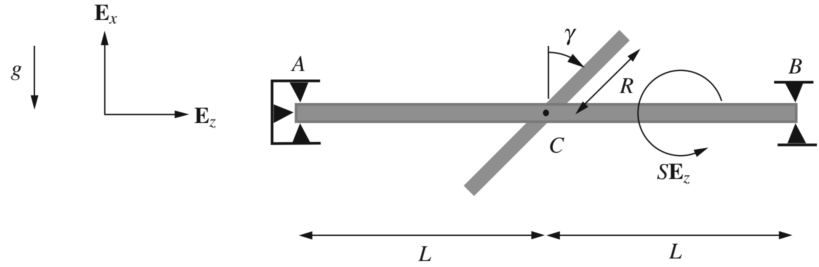

Fig. 9.12 *The imbalanced rotor problem.*

the other hand, when $\beta \tan(\phi) = \mu_d = \mu_s$, then, as shown in Figure 9.11(b), sliding with $v_P > 0$ persists indefinitely, and if v_P is initially negative then a transition to rolling will eventually occur. Finally, if there is insufficient friction to ensure that rolling can occur, then sliding persists and v_P will increase indefinitely (see Figure 9.11(c)). That is, the friction force is unable to overcome the gravitational force and prevent the center of mass from continually accelerating.

9.7 An Imbalanced Rotor

Many applications can be modeled as a rigid body rotating about a fixed axis. In particular, driveshafts in automobiles and turbomachinery. If the rigid body is balanced, then the axis of rotation corresponds to a principal axis of the rigid body. For example, if $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$, then $\mathbf{H} = I_{zz}\dot{\theta}\mathbf{E}_z$. However, this is very difficult and expensive to achieve. An imbalance is said to occur in situations where either or both of I_{xz} and/or I_{yz} are nonzero. In this case, $\mathbf{H}$ and $\boldsymbol{\omega}$ are not parallel.

The imbalance is a common feature of wheel assemblies on cars, and it is typical to add weights, which usually range from 0.01 kilograms to 0.15 kilograms, onto the rims of the wheels to reduce this imbalance. Special machines, known as wheel balancers, are used which spin the wheels at high speeds so that the imbalance can be detected and the appropriate location for the added weights determined. As we show shortly, in rotating systems, such as wheel assemblies or a rotating shaft, imbalance manifests in bearing forces that are periodic functions of time and are ultimately destructive to the bearings. Furthermore, the magnitude of these oscillatory bearing forces grows quadratically with the rotational speed. Thus, and as is also observed in practice, the effects of the imbalance on the bearings in cars with an imbalanced wheel become more pronounced and destructive as the car's speed increases.

To illustrate the above phenomena, we turn to discussing the prototypical example of an imbalanced rigid body shown in Figure 9.12. It consists of a homogeneous

mer nears the pins (see Frohlich [29]). Some readers may also have noticed the same phenomenon in pool, a problem that is discussed in Article 239 of Routh [65].

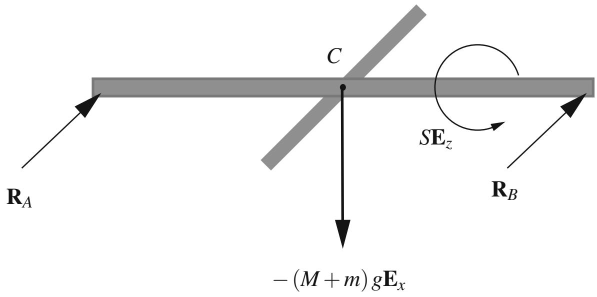

Fig. 9.13 *Free-body diagram of the rotating shaft.*

disk of mass m, radius R, and thickness h that is welded to a homogeneous shaft of mass M, length $2L$, and radius r. The centers of mass of these rigid bodies are coincident, and the disk is inclined at an angle γ to the vertical. The rigid body, which consists of the shaft and the disk, is supported by bearings at A and B. This rigid body is often called a rotor. Finally, an applied torque $S\mathbf{E}_z$ acts on the shaft.

9.7.1 Kinematics

We assume that the motion is such that the axis of rotation is fixed: $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z = \omega\mathbf{E}_z$ and $\boldsymbol{\alpha} = \ddot{\theta}\mathbf{E}_z$. The corotational basis is defined in the usual manner:

$$\mathbf{e}_x = \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \qquad \mathbf{e}_y = -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y, \qquad \mathbf{e}_z = \mathbf{E}_z.$$

We also recall the relations

$$\dot{\mathbf{e}}_x = \dot{\theta}\mathbf{e}_y, \qquad \dot{\mathbf{e}}_y = -\dot{\theta}\mathbf{e}_x, \qquad \dot{\mathbf{e}}_z = \mathbf{0}.$$

We denote the position vector of the center of mass of the rigid body by

$$\bar{\mathbf{x}} = x_0\mathbf{E}_x + y_0\mathbf{E}_y + z_0\mathbf{E}_z.$$

Here, the bearings at A and B are such that $\bar{\mathbf{x}}$ is a constant:

$$\bar{\mathbf{v}} = \mathbf{0}, \qquad \bar{\mathbf{a}} = \mathbf{0}.$$

Next, we address the angular momentum of the rigid body. Here, it is important to note that $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are not principal axes. For a rigid body rotating about the $\mathbf{e}_z$-axis, we recall from Section 9.1.3 that

$$\begin{aligned}
\mathbf{H} &= I_{xz}\omega\mathbf{e}_x + I_{yz}\omega\mathbf{e}_y + I_{zz}\omega\mathbf{E}_z, \\
\dot{\mathbf{H}} &= \left(I_{xz}\dot{\omega} - I_{yz}\omega^2\right)\mathbf{e}_x + \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{E}_z.
\end{aligned}$$

For the rigid body of interest, a long, but straightforward, calculation shows that the inertias of interest are

$$
\begin{aligned}
I_{xz} &= \left(\frac{mh^2}{12} - \frac{mR^2}{4}\right)\sin(\gamma)\cos(\gamma),\\
I_{yz} &= 0,\\
I_{zz} &= \frac{Mr^2}{2} + \left(\frac{mR^2}{4} + \frac{mh^2}{12}\right)\sin^2(\gamma) + \frac{mR^2}{2}\cos^2(\gamma).
\end{aligned}
$$

Notice that if $\gamma = 0°$, $90°$, $180°$, or $270°$, then I_{xz} vanishes and $\mathbf{E}_z$ is a principal axis of the rigid body.

9.7.2 Forces and Moments

The free-body diagram is shown in Figure 9.13. The reaction forces at the bearings have the representations

$$
\mathbf{R}_A = R_{A_x}\mathbf{E}_x + R_{A_y}\mathbf{E}_y + R_{A_z}\mathbf{E}_z, \qquad \mathbf{R}_B = R_{B_x}\mathbf{E}_x + R_{B_y}\mathbf{E}_y.
$$

These five forces ensure that the center of mass remains fixed and that the angular velocity vector has components in the $\mathbf{E}_z$ direction only. There is also a gravitational force on the rigid body, $-(M+m)g\mathbf{E}_x$, in addition to an applied torque $S\mathbf{E}_z$. Consequently, the resultant force and moment acting on the rigid body are

$$
\mathbf{F} = \mathbf{R}_A + \mathbf{R}_B - (M+m)g\mathbf{E}_x, \qquad \mathbf{M} = L\mathbf{E}_z \times (\mathbf{R}_B - \mathbf{R}_A) + S\mathbf{E}_z.
$$

9.7.3 Balance Laws

From the balance laws, we find that

$$
\begin{aligned}
\mathbf{R}_A + \mathbf{R}_B - (M+m)g\mathbf{E}_x &= \mathbf{0},\\
L\mathbf{E}_z \times (\mathbf{R}_B - \mathbf{R}_A) + S\mathbf{E}_z &= I_{xz}\dot{\omega}\mathbf{e}_x + I_{xz}\omega^2\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{E}_z.
\end{aligned}
$$

As a result, we have six equations for the five unknown components of the reaction forces and a differential equation for θ.

9.7.4 Analysis

We turn our attention to the case where $\boldsymbol{\omega}$ is a constant: $\boldsymbol{\omega} = \omega_0 \mathbf{E}_z$. The six equations above simplify to

$$R_{A_x} + R_{B_x} - (M+m)g = 0, \qquad R_{A_y} + R_{B_y} = 0, \qquad R_{A_z} = 0,$$

$$L\left(R_{A_y} - R_{B_y}\right) = -I_{xz}\omega_0^2 \sin(\omega_0 t), \quad L\left(R_{B_x} - R_{A_x}\right) = I_{xz}\omega_0^2 \cos(\omega_0 t), \quad S = 0.$$

Notice that from the $\mathbf{E}_z$ component of the balance of angular momentum we find that $S = 0$. That is, no applied torque is needed to rotate the rigid body at a constant angular speed. Solving for the bearing forces, we obtain

$$\mathbf{R}_A = \left(\frac{M+m}{2}\right) g\mathbf{E}_x - \left(\frac{I_{xz}\omega_0^2}{2L}\right)\left(\cos(\omega_0 t)\mathbf{E}_x + \sin(\omega_0 t)\mathbf{E}_y\right),$$
$$\mathbf{R}_B = \left(\frac{M+m}{2}\right) g\mathbf{E}_x + \left(\frac{I_{xz}\omega_0^2}{2L}\right)\left(\cos(\omega_0 t)\mathbf{E}_x + \sin(\omega_0 t)\mathbf{E}_y\right).$$

Clearly, these forces are the superposition of constant and periodic terms. Recalling the expression for I_{xz} given previously, it is easy to see that the periodic component of these forces vanishes when $\gamma = 0°$, $90°$, $180°$, or $270°$. The rigid body is then said to be balanced.

To illustrate the features of bearing forces in imbalanced situations that we mentioned earlier in this section, we now consider the component R_{B_x} of $\mathbf{R}_B$. As shown in Figure 9.14, when the rotor is stationary, R_{B_x} is constant. However, as the speed ω_0 increases, the magnitude of the oscillatory component of R_{B_x} grows in magnitude quadratically in ω_0.

9.8 Summary

The first set of important results in this chapter pertained to a system of forces and a pure moment acting on a rigid body. Specifically, for a system of K forces $\mathbf{F}_i$ $(i = 1, \ldots, K)$ and a moment $\mathbf{M}_p$, which is not due to the moment of an applied force, acting on the rigid body, the resultant force $\mathbf{F}$ and moments are

$$\mathbf{F} = \sum_{i=1}^{K} \mathbf{F}_i,$$

$$\mathbf{M}_O = \mathbf{M}_p + \sum_{i=1}^{K} \mathbf{x}_i \times \mathbf{F}_i,$$

$$\mathbf{M} = \mathbf{M}_p + \sum_{i=1}^{K} (\mathbf{x}_i - \bar{\mathbf{x}}) \times \mathbf{F}_i.$$

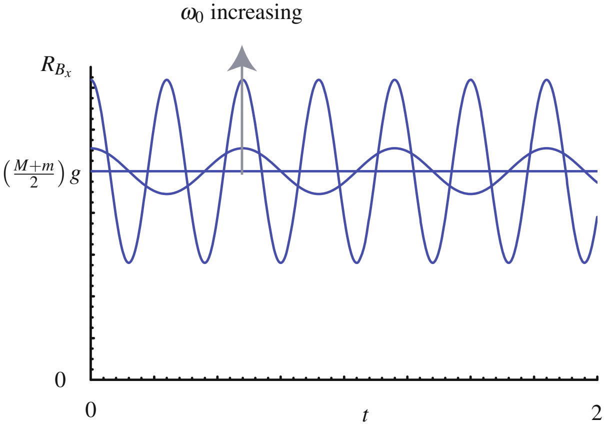

Fig. 9.14 *The component R_{B_x} of the bearing force $\mathbf{R}_B$ for the imbalanced rotor as a function of the rotation speed. In this figure, the three cases corresponding to a stationary rotor, a rotor spinning at 100 RPM, and a rotor spinning at 200 RPM are shown.*

Here, $\mathbf{M}_O$ is the resultant moment relative to a fixed point O and $\mathbf{M}$ is the resultant moment relative to the center of mass C of the rigid body.

The relationship between forces and moments and the motion of the rigid body is postulated using the balance laws. There are two equivalent sets of balance laws:

$$\mathbf{F} = m\dot{\bar{\mathbf{v}}}, \qquad \mathbf{M}_O = \dot{\mathbf{H}}_O,$$

and

$$\mathbf{F} = m\dot{\bar{\mathbf{v}}}, \qquad \mathbf{M} = \dot{\mathbf{H}}.$$

When these balance laws are specialized to the case of a fixed-axis rotation, the expressions for $\dot{\mathbf{H}}_O$ and $\dot{\mathbf{H}}$ simplify. For instance,

$$\begin{aligned}\mathbf{F} &= m\dot{\bar{\mathbf{v}}},\\ \mathbf{M} &= \left(I_{xz}\dot{\omega} - I_{yz}\omega^2\right)\mathbf{e}_x + \left(I_{yz}\dot{\omega} + I_{xz}\omega^2\right)\mathbf{e}_y + I_{zz}\dot{\omega}\mathbf{E}_z.\end{aligned}$$

In most problems, $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$ are chosen such that $I_{xz} = I_{yz} = 0$.

To establish conservations of energy, two equivalent forms of the work-energy theorem were developed in Section 9.2. First, however, the Koenig decomposition for the kinetic energy of a rigid body was established:

$$T = \frac{1}{2}m\bar{\mathbf{v}}\cdot\bar{\mathbf{v}} + \frac{1}{2}\mathbf{H}\cdot\boldsymbol{\omega}.$$

This was then followed by a development of the work-energy theorem for a rigid body:

$$\frac{dT}{dt} = \mathbf{F}\cdot\bar{\mathbf{v}} + \mathbf{M}\cdot\boldsymbol{\omega} = \sum_{i=1}^{K}\mathbf{F}_i\cdot\mathbf{v}_i + \mathbf{M}_p\cdot\boldsymbol{\omega}.$$

To establish energy conservation results, this theorem is used in a similar manner to the one employed with particles and systems of particles.

Four sets of applications were then discussed:

1. Purely translational motion of a rigid body where $\boldsymbol{\omega} = \boldsymbol{\alpha} = \mathbf{0}$.
2. A rigid body with a fixed point O.
3. Rolling rigid bodies and sliding rigid bodies.
4. Imbalanced rotors.

It is important to note that for the second set of applications, the balance law $\mathbf{M}_O = \dot{\mathbf{H}}_O$ is more convenient to use than $\mathbf{M} = \dot{\mathbf{H}}$. The role of $\mathbf{M}_c$ in these problems is to ensure that the axis of rotation remains $\mathbf{E}_z$. Finally, the four steps discussed in Section 9.1.4 are used as a guide to solving all of the applications.

9.9 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

9.1. Starting from the definitions of $\mathbf{M}$ and $\mathbf{M}_O$, show that $\mathbf{M}_O = \dot{\mathbf{H}}_O$ along with $\mathbf{F} = m\dot{\bar{\mathbf{v}}}$ implies that $\mathbf{M} = \dot{\mathbf{H}}$.

9.2. For the overturning cart discussed in Section 9.3.1, show that if the cart is tall (i.e., $a \gg b$) then the toppling force P is smaller than if the cart were stout (i.e., $a \ll b$).

9.3. Consider a cart with the same dimensions as the one discussed in Section 9.3.1. Suppose that the applied force $\mathbf{P} = \mathbf{0}$, but the front wheels are driven. The driving force on the respective front wheels is assumed to be

$$\mathbf{F}_2 = \mu N_{2_y} \mathbf{E}_x, \qquad \mathbf{F}_3 = \mu N_{3_y} \mathbf{E}_x,$$

where μ is a constant. Calculate the resulting acceleration vector of the center of mass of the cart.

9.4. Consider the example of a rigid body rotating about a fixed point O discussed in Section 9.4. Starting from $\mathbf{H}_O = \mathbf{H} + \bar{\mathbf{x}} \times m\bar{\mathbf{v}}$ and using the identity $\bar{\mathbf{v}} = \boldsymbol{\omega} \times \bar{\mathbf{x}}$ show that

$$\mathbf{H}_O \cdot \boldsymbol{\omega} = \mathbf{H} \cdot \boldsymbol{\omega} + m\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}.$$

Why is this result useful?

9.5. When solving the example discussed in Section 9.4, one person uses the balance law $\mathbf{M} = \dot{\mathbf{H}}$ instead of $\mathbf{M}_O = \dot{\mathbf{H}}_O$. Why is that approach valid?

9.6. As a special case of the example discussed in Section 9.4, consider a long slender rod of length L and mass m that is attached at one of its ends to a pin-joint. Show that

$$\mathbf{H} = \frac{1}{12} mL^2 \dot{\theta} \mathbf{E}_z, \qquad \bar{\mathbf{v}} = \frac{L}{2} \dot{\theta} \mathbf{e}_y, \qquad \mathbf{H}_O = \frac{1}{3} mL^2 \dot{\theta} \mathbf{E}_z.$$

9.7. For the problem discussed in Exercise 9.6, show that if the sole applied force acting on the rod is a gravitational force, $-mg\mathbf{E}_y$, the equation governing the motion of the rod is

$$\frac{mL^2}{3}\ddot{\theta} = -\frac{mgL}{2}\cos(\theta).$$

Furthermore, prove that the total energy of the rod is conserved. Why is this problem analogous to a planar pendulum problem?

9.8. Consider a rigid body rolling on a fixed surface. Suppose that, apart from the friction and normal forces at the point of contact P, the applied forces acting on the body are conservative; then why is the total energy of the rigid body conserved?

9.9. Using the static friction criterion discussed in Section 9.6.3, show that a circular disk of mass m and radius R can roll without slipping on a steeper incline than a circular hoop of the same mass and radius.

9.10. For a rolling disk on a horizontal incline, show that $|\ddot{x}| \leq \mu_s g$. Similarly, for a sliding disk show that $|\ddot{x}| = \mu_d g$. Using these results explain why rolling disks decelerate faster than sliding disks. This observation is the reason for the desirability of anti-lock braking systems (ABS) in automobiles.

9.11. Consider the sliding rigid disk discussed in Section 9.6.4 and suppose that $\phi = 0$. Determine $\theta(t)$ and $x(t)$ for the two cases where initially $\text{sgn}(v_P) > 0$ and $\text{sgn}(v_P) < 0$.

9.12. Recall the imbalanced rotor discussed in Section 9.7. For various values of ω_0, plot the components R_{A_x}, R_{B_x}, R_{A_y}, and R_{B_y} using the numerical values $L = 10$ meters, $m = 20$ kilograms, $M = 100$ kilograms, $\gamma = -0.01$ radians, $h = 0.01$ meters, and $R = 1$ meter.

Part IV
Dynamics of Systems of Particles and Rigid Bodies

Chapter 10
Systems of Particles and Rigid Bodies

TOPICS

This chapter is the culmination of the primer. To start, the linear momentum of a system of K particles and N rigid bodies is discussed. Similarly, the angular momenta and kinetic energy of such a system are developed. We then turn to the balance laws for such a system. The complete analysis of the resulting differential equations that these laws provide is usually beyond the scope of an undergraduate engineering dynamics course, and instead we focus on some particular results. These results involve using conservations of energy, angular momentum, and linear momentum. We then illustrate how one or more such conservations can be used to obtain solutions to some problems.

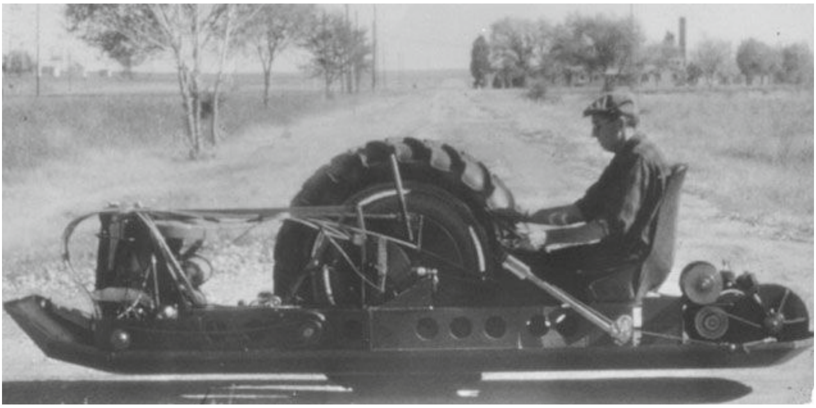

Fig. 10.1 *The American inventor Charles F. Taylor (1916–1997) on a prototype of his one-wheeled vehicle in the early 1960s in Colorado.*

O.M. O'Reilly, *Engineering Dynamics: A Primer*, DOI 10.1007/978-1-4419-6360-4_10,

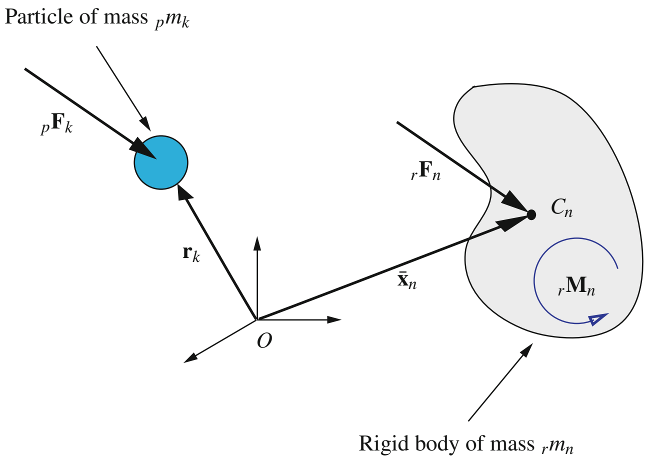

Fig. 10.2 *The kth particle and the nth rigid body.*

10.1 A System of Particles and Rigid Bodies

The systems of interest feature one or more rigid bodies. An example of such a system is shown in Figure 10.1. This one-wheeled vehicle was developed in the 1950s and 1960s by Charles Taylor. It features a platform-mounted single large wheel which is driven by a motor. The vehicle is stabilized and steered by gyroscopes. A simple model for this system, which features two coupled rigid bodies, in discussed in [44]. Analyzing the dynamics of such a model is dependent on first establishing expressions for the momenta and kinetic energies of its components. In this section, we analyze systems of rigid bodies and particles with the aim of being able to establish expressions for their energies and momenta. To cover as many cases as possible, we now discuss a particular system of K particles and N rigid bodies. It is sufficiently general to cater to all of our subsequent developments and examples.

We use the index k to identify a specific particle $(k = 1, \ldots, K)$. The mass of the kth particle is denoted by $_pm_k$, its position vector relative to a fixed origin O by $\mathbf{r}_k$, and the resultant external force acting on the particle is denoted by $_p\mathbf{F}_k$ (see Figure 10.2).

Similarly, we use the index n to identify a specific rigid body $(n = 1, \ldots, N)$. The mass of the nth rigid body is denoted by $_rm_n$, the position vector of its center of mass C_n relative to a fixed origin O by $\bar{\mathbf{x}}_n$, its angular velocity vector by $\boldsymbol{\omega}_n$, and the resultant external force and moment (relative to its center of mass) acting on the rigid body are denoted by $_r\mathbf{F}_n$ and $_r\mathbf{M}_n$, respectively (see Figure 10.2).

The center of mass C of this system has a position vector $\bar{\mathbf{x}}$, which is defined by

$$\bar{\mathbf{x}} = \frac{1}{m}\left(\sum_{k=1}^{K} {}_pm_k\mathbf{r}_k + \sum_{n=1}^{N} {}_rm_n\bar{\mathbf{x}}_n\right),$$

where m is the total mass of the system:

$$m = \sum_{k=1}^{K} {}_{p}m_k + \sum_{n=1}^{N} {}_{r}m_n.$$

You should notice how this definition is an obvious extension to others that you have seen previously.

10.1.1 Momenta and Kinetic Energy

The linear momentum $\mathbf{G}$ of the system is the sum of the linear momenta of the individual particles and rigid bodies:

$$\mathbf{G} = \sum_{k=1}^{K} {}_{p}m_k \dot{\mathbf{r}}_k + \sum_{n=1}^{N} {}_{r}m_n \dot{\bar{\mathbf{x}}}_n.$$

Notice that the linear momentum of the system can also be expressed in terms of the velocity vector of the center of mass of the system and the total mass of the system:

$$\mathbf{G} = m\dot{\bar{\mathbf{x}}}.$$

This result follows from the definition of the center of mass.

The angular momentum $\mathbf{H}_O$ of the system relative to the fixed point O is the sum of the individual angular momenta relative to O:

$$\mathbf{H}_O = \sum_{k=1}^{K} \mathbf{r}_k \times {}_{p}m_k \dot{\mathbf{r}}_k + \sum_{n=1}^{N} \left(\mathbf{H}_n + \bar{\mathbf{x}}_n \times {}_{r}m_n \dot{\bar{\mathbf{x}}}_n\right).$$

Similarly, the angular momentum $\mathbf{H}$ of the system relative to its center of mass is

$$\mathbf{H} = \sum_{k=1}^{K} (\mathbf{r}_k - \bar{\mathbf{x}}) \times {}_{p}m_k \dot{\mathbf{r}}_k + \sum_{n=1}^{N} \left(\mathbf{H}_n + (\bar{\mathbf{x}}_n - \bar{\mathbf{x}}) \times {}_{r}m_n \dot{\bar{\mathbf{x}}}_n\right).$$

In both of the previous equations, $\mathbf{H}_n$ is the angular momentum of the nth rigid body relative to its center of mass. You should notice that we have used the identities

$$\mathbf{H}_{O_n} = \mathbf{H}_n + \bar{\mathbf{x}}_n \times {}_{r}m_n \dot{\bar{\mathbf{x}}}_n,$$

where $\mathbf{H}_{O_n}$ is the angular momentum of the nth rigid body relative to O.

Finally, the kinetic energy T of the system is defined to be the sum of the kinetic energies of its constituents:

$$T = \sum_{k=1}^{K} \frac{1}{2} {}_{p}m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k + \sum_{n=1}^{N} \left(\frac{1}{2} {}_{r}m_n \dot{\bar{\mathbf{x}}}_n \cdot \dot{\bar{\mathbf{x}}}_n + \frac{1}{2} \mathbf{H}_n \cdot \boldsymbol{\omega}_n\right).$$

10.1.2 Impulses, Momenta, and Balance Laws

For each rigid body of the system, one has the balance of linear momentum and the balance of angular momentum:

$$ {}_r\mathbf{F}_n = {}_r m_n \ddot{\bar{\mathbf{x}}}_n, \qquad \mathbf{M}_{O_n} = \dot{\mathbf{H}}_{O_n} \qquad (n = 1, \ldots, N). $$

Here, $\mathbf{M}_{O_n}$ is the resultant moment, relative to the point O, acting on the nth rigid body. We also have an alternative form of the balance of angular momentum relative to the center of mass of the nth rigid body: $\mathbf{M}_n = \dot{\mathbf{H}}_n$. In addition, for each particle one has the balance of linear momentum:

$$ {}_p\mathbf{F}_k = {}_p m_k \ddot{\mathbf{r}}_k \qquad (k = 1, \ldots, K). $$

As discussed in Chapter 6, the angular momentum theorem for a particle is derived from this balance law. Hence, for a particle, the balance of angular momentum is not a separate postulate as it is with rigid bodies.

Let $\mathbf{M}_O$ denote the resultant moment acting on the system relative to the point O, $\mathbf{M}$ denote the resultant moment acting on the system relative to its center of mass, and $\mathbf{F}$ denote the resultant force acting on the system. Then, adding the balances of linear momenta, we find that

$$ \mathbf{F} = \dot{\mathbf{G}}, $$

where

$$ \mathbf{F} = \sum_{k=1}^{K} {}_p\mathbf{F}_k + \sum_{n=1}^{N} {}_r\mathbf{F}_n. $$

Similarly, adding the balances of angular momenta relative to the point O,

$$ \mathbf{M}_O = \dot{\mathbf{H}}_O, $$

where

$$ \mathbf{M}_O = \sum_{k=1}^{K} \left({}_p\mathbf{r}_k \times {}_p\mathbf{F}_k\right) + \sum_{n=1}^{N} \mathbf{M}_{O_n}. $$

Hence, we have balances of linear and angular momenta for the system.[1]

Another form of the balance laws can be obtained by integrating both sides of $\mathbf{F} = \dot{\mathbf{G}}$, $\mathbf{M} = \dot{\mathbf{H}}$, and $\mathbf{M}_O = \dot{\mathbf{H}}_O$. These forms are known as the impulse-momentum

[1] You should be able to consider special cases of these results: for example, cases where the system of interest contains either no rigid bodies or no particles. Additionally, we could establish a balance of angular momentum relative to the center of mass of the system ($\mathbf{M} = \dot{\mathbf{H}}$), but we leave this as an exercise. Such an exercise involves using $\mathbf{F} = \dot{\mathbf{G}}$ and $\mathbf{M}_O = \dot{\mathbf{H}}_O$. Its proof is similar to that used to establish the corresponding result for a single rigid body.

forms or the integral forms of the balance laws:

$$\mathbf{G}(t) - \mathbf{G}(t_0) = \int_{t_0}^{t} \mathbf{F}(\tau) d\tau,$$
$$\mathbf{H}(t) - \mathbf{H}(t_0) = \int_{t_0}^{t} \mathbf{M}(\tau) d\tau,$$
$$\mathbf{H}_O(t) - \mathbf{H}_O(t_0) = \int_{t_0}^{t} \mathbf{M}_O(\tau) d\tau.$$

Here, the integral of the force $\mathbf{F}$ is known as the *linear impulse* (of $\mathbf{F}$), and the integral of the moment $\mathbf{M}_O$ is known as the *angular impulse* (of $\mathbf{M}_O$). As you have already witnessed in Chapter 6, these forms of the balance laws can be extremely useful in analyzing impact problems, where the impulse of certain forces and moments dominates contributions from other forces and moments.

Although we did not discuss the impulse-momentum forms of the balance laws for rigid bodies, it should be obvious that for each rigid body, one has

$$_r m_n \bar{\mathbf{v}}_n(t) - {_r m_n} \bar{\mathbf{v}}_n(t_0) = \int_{t_0}^{t} {_r\mathbf{F}_n}(\tau) d\tau,$$
$$\mathbf{H}_{O_n}(t) - \mathbf{H}_{O_n}(t_0) = \int_{t_0}^{t} \mathbf{M}_{O_n}(\tau) d\tau \qquad (n = 1, \ldots, N).$$

The corresponding results for a single particle were discussed in Chapter 6.

10.1.3 Conservations

We now address the possibility that in certain problems, a component of the linear momentum $\mathbf{G}$ of the system, a component of the angular momenta $\mathbf{H}_O$ or $\mathbf{H}$ of the system, and/or the total energy E of the system is conserved. We have developed these results three times previously: once for a single particle, once for a system of particles, and once for a single rigid body.

10.1.3.1 Conservation of Linear Momentum

Let us first deal with conservation of linear momentum. Given a vector $\mathbf{c} = \mathbf{c}(t)$, then it is easily seen that for $\mathbf{G} \cdot \mathbf{c}$ to be constant during the motion of the system, we must have $\mathbf{F} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = 0$. Finding the vector $\mathbf{c}$ that satisfies $\mathbf{F} \cdot \mathbf{c} + \mathbf{G} \cdot \dot{\mathbf{c}} = 0$ for a particular system is an art, and we discuss examples shortly. You should notice that if $\mathbf{F} = \mathbf{0}$, then $\mathbf{G}$ is conserved. The conservation of components of $\mathbf{G}$ is often used in impact problems.

10.1.3.2 Conservation of Angular Momentum

Next, we have conservation of angular momentum. Given a vector $\mathbf{c} = \mathbf{c}(t)$, we seek to determine when $\mathbf{H}_O \cdot \mathbf{c}$ is constant during the motion of the system. To do this, we calculate

$$\frac{d}{dt}(\mathbf{H}_O \cdot \mathbf{c}) = \dot{\mathbf{H}}_O \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = \mathbf{M}_O \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}}.$$

It follows that it is necessary and sufficient for $\mathbf{M}_O \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = 0$ for $\mathbf{H}_O \cdot \mathbf{c}$ to be conserved during the motion of the system. Again, finding $\mathbf{c}$ such that $\mathbf{M}_O \cdot \mathbf{c} + \mathbf{H}_O \cdot \dot{\mathbf{c}} = 0$ for a given system is an art. You should notice that if $\mathbf{M}_O = \mathbf{0}$, then $\mathbf{H}_O$ is conserved. The corresponding results for $\mathbf{H}$ are easily inferred.

The most common occurrence of conservation of angular momentum is a system of interconnected rigid bodies in space. There, because one assumes that $\mathbf{M} = \mathbf{0}$, the angular momentum $\mathbf{H}$ is conserved, and astronauts use this conservation to change their orientation during space walks. Specifically, one can consider an astronaut as a system of rigid bodies. By changing the relative orientation of these bodies, they change the angular velocity vectors of parts of their bodies and in this manner change their overall orientation. A similar principle is behind the falling cat, which "always" seems to land on its feet.[2]

10.1.3.3 Conservation of Energy

Finally, we turn to the conservation of energy. We start with the definition of the kinetic energy of the system and, by using the work-energy theorems for the individual particles and rigid bodies, we establish a work-energy theorem for the system. This work-energy theorem was used, as in the cases of single particles, systems of particles, and single rigid bodies, to establish whether the total energy of the system is conserved. We now proceed to establish the work-energy theorem for the system.

Recall that

$$T = \sum_{k=1}^{K} \frac{1}{2} {}_p m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k + \sum_{n=1}^{N} \left(\frac{1}{2} {}_r m_n \dot{\bar{\mathbf{x}}}_n \cdot \dot{\bar{\mathbf{x}}}_n + \frac{1}{2} \mathbf{H}_n \cdot \boldsymbol{\omega}_n \right).$$

Taking the derivative of this expression and using the work-energy theorems discussed previously,

$$\frac{d}{dt}\left(\frac{1}{2} {}_p m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k \right) = {}_p\mathbf{F}_k \cdot \dot{\mathbf{r}}_k \qquad (k = 1, \ldots, K),$$

$$\frac{d}{dt}\left(\frac{1}{2} {}_r m_n \dot{\bar{\mathbf{x}}}_n \cdot \dot{\bar{\mathbf{x}}}_n + \frac{1}{2} \mathbf{H}_n \cdot \boldsymbol{\omega}_n \right) = {}_r\mathbf{F}_n \cdot \dot{\bar{\mathbf{x}}}_n + \mathbf{M}_n \cdot \boldsymbol{\omega}_n \qquad (n = 1, \ldots, N),$$

[2] See the photos for the falling cat in Crabtree [19] and Kane and Scher [40]. References to modern approaches to this problem can be found in Fecko [27] and Shapere and Wilczek [70].

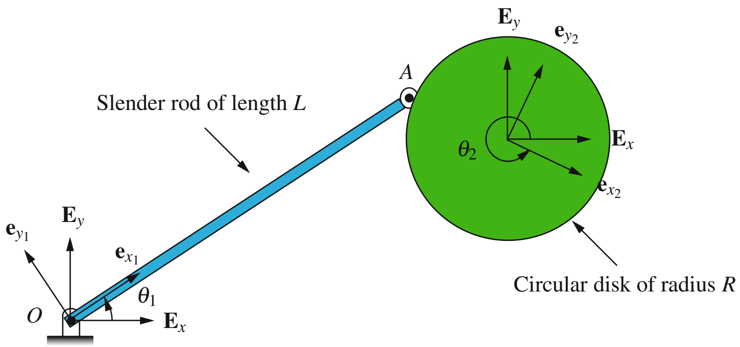

Fig. 10.3 *Two coupled rigid bodies.*

we find that

$$\frac{dT}{dt} = \sum_{k=1}^{K} {}_p\mathbf{F}_k \cdot \dot{\mathbf{r}}_k + \sum_{n=1}^{N} \left({}_r\mathbf{F}_n \cdot \dot{\bar{\mathbf{x}}}_n + \mathbf{M}_n \cdot \boldsymbol{\omega}_n \right).$$

This is the work-energy theorem for the system. Starting from this theorem and substituting for the forces and moments on the individual constituents of the system, one can ascertain whether the total energy of the system is conserved. Again, this procedure is identical to the one previously discussed. You should also note that for each rigid body we have an alternative form of the work-energy theorem that we can use in place of the terms on the right-hand side of the above equation.[3] We do this in the example below.

10.2 An Example of Two Rigid Bodies

Here, as our first example, we consider two connected rigid bodies (cf. Figure 10.3). One of the bodies is a slender rod of length L and mass m_1 that is pin-jointed at O. This point is fixed. The other is a circular disk of mass m_2 and radius R. They are connected by a pin-joint at the point A that lies at the outer extremity of the rod and the edge of the disk. Both bodies rotate about the $\mathbf{E}_z$ axis, and this axis is also a principal axis for both bodies. A gravitational force acts on each of the bodies.

We wish to determine which kinematical quantities are conserved for this system. At the end of this section we discuss related problems and questions concerning this system.

---

[3] See Section 9.2.3 of Chapter 9.

10.2.1 Kinematics

To proceed, we first calculate the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_O$, and kinetic energy T of the system. It is convenient to define two corotational bases $\left\{\mathbf{e}_{x_1}, \mathbf{e}_{y_1}, \mathbf{e}_{z_1} = \mathbf{E}_z\right\}$ and $\left\{\mathbf{e}_{x_2}, \mathbf{e}_{y_2}, \mathbf{e}_{z_2} = \mathbf{E}_z\right\}$. You should notice that these bases are related by

$$\begin{aligned}\mathbf{e}_{x_2} &= \cos(\theta_2 - \theta_1)\,\mathbf{e}_{x_1} + \sin(\theta_2 - \theta_1)\,\mathbf{e}_{y_1},\\ \mathbf{e}_{y_2} &= \cos(\theta_2 - \theta_1)\,\mathbf{e}_{y_1} - \sin(\theta_2 - \theta_1)\,\mathbf{e}_{x_1}.\end{aligned}$$

Because $\mathbf{E}_z$ is a principal axis, we can use our results from Section 8.8 of Chapter 8 to write

$$\mathbf{H}_1 = \frac{m_1 L^2}{12}\dot{\theta}_1 \mathbf{E}_z, \qquad \mathbf{H}_2 = \frac{m_2 R^2}{2}\dot{\theta}_2 \mathbf{E}_z.$$

You should notice that the angular velocity vectors of the bodies are

$$\boldsymbol{\omega}_1 = \dot{\theta}_1 \mathbf{E}_z, \qquad \boldsymbol{\omega}_2 = \dot{\theta}_2 \mathbf{E}_z.$$

These velocities were used to calculate the angular momenta of the bodies relative to their centers of mass.

One also has the following representations:[4]

$$\bar{\mathbf{x}}_1 = \frac{L}{2}\mathbf{e}_{x_1}, \qquad \bar{\mathbf{x}}_2 = L\mathbf{e}_{x_1} + R\mathbf{e}_{x_2}, \qquad \mathbf{x}_A = L\mathbf{e}_{x_1}.$$

Differentiating these representations, we find that

$$\bar{\mathbf{v}}_1 = \frac{L}{2}\dot{\theta}_1 \mathbf{e}_{y_1}, \qquad \bar{\mathbf{v}}_2 = L\dot{\theta}_1 \mathbf{e}_{y_1} + R\dot{\theta}_2 \mathbf{e}_{y_2}, \qquad \mathbf{v}_A = L\dot{\theta}_1 \mathbf{e}_{y_1}.$$

Hence, the linear momentum of the system is

$$\mathbf{G} = (m_1 + 2m_2)\frac{L}{2}\dot{\theta}_1 \mathbf{e}_{y_1} + m_2 R\dot{\theta}_2 \mathbf{e}_{y_2}.$$

The angular momentum of the system relative to O is, by definition,

$$\mathbf{H}_O = \mathbf{H}_1 + \bar{\mathbf{x}}_1 \times m_1 \bar{\mathbf{v}}_1 + \mathbf{H}_2 + \bar{\mathbf{x}}_2 \times m_2 \bar{\mathbf{v}}_2.$$

Substituting for the kinematical quantities on the right-hand side of this equation, one obtains, after a substantial amount of algebra,

$$\mathbf{H}_O = \frac{m_1 L^2}{3}\dot{\theta}_1 \mathbf{E}_z + m_2\left(L^2\dot{\theta}_1 + \frac{3R^2}{2}\dot{\theta}_2\right)\mathbf{E}_z + m_2 RL\left(\dot{\theta}_2 + \dot{\theta}_1\right)\cos(\theta_2 - \theta_1)\,\mathbf{E}_z.$$

[4] For ease of notation, we drop the subscripts r and p used in the previous sections.

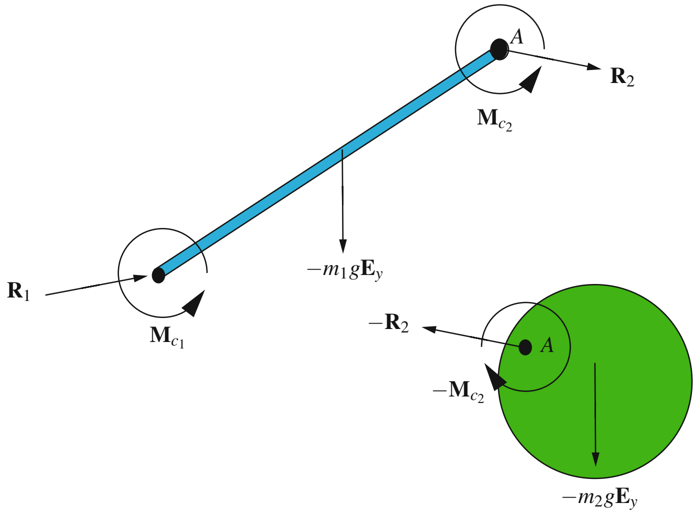

Fig. 10.4 *Free-body diagrams of the rod and the disk.*

Finally, the kinetic energy of the system is

$$T = \frac{1}{2}m_1\bar{\mathbf{v}}_1 \cdot \bar{\mathbf{v}}_1 + \frac{1}{2}\mathbf{H}_1 \cdot \boldsymbol{\omega}_1 + \frac{1}{2}m_2\bar{\mathbf{v}}_2 \cdot \bar{\mathbf{v}}_2 + \frac{1}{2}\mathbf{H}_2 \cdot \boldsymbol{\omega}_2.$$

All of the ingredients are present to write this expression in terms of the kinematical quantities discussed earlier:

$$T = \frac{m_1L^2}{6}\dot{\theta}_1^2 + \frac{m_2}{2}\left(L^2\dot{\theta}_1^2 + \frac{3R^2}{2}\dot{\theta}_2^2 + 2RL\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_2 - \theta_1\right)\right).$$

10.2.2 Forces and Moments

In order to examine which conserved quantities are present in this system, we first need to determine the forces and moments acting on each body. These are summarized in the free-body diagrams shown in Figure 10.4. You should notice that there is a reaction force $\mathbf{R}_1 = R_{1_x}\mathbf{E}_x + R_{1_y}\mathbf{E}_y + R_{1_z}\mathbf{E}_z$ at O, equal and opposite reaction forces of the form $\mathbf{R}_2 = R_{2_x}\mathbf{E}_x + R_{2_y}\mathbf{E}_y + R_{2_z}\mathbf{E}_z$ at A, a reaction moment at O, $\mathbf{M}_{c_1} = M_{c_{1_x}}\mathbf{E}_x + M_{c_{1_y}}\mathbf{E}_y$, and equal and opposite reaction moments of the form $\mathbf{M}_{c_2} = M_{c_{2_x}}\mathbf{E}_x + M_{c_{2_y}}\mathbf{E}_y$ at A. These reactions ensure that the bodies are connected and their angular velocities are in the $\mathbf{E}_z$ direction.

In summary, the resultant forces are

$$\begin{aligned}\mathbf{F}_1 &= \mathbf{R}_1 + \mathbf{R}_2 - m_1 g \mathbf{E}_y,\\ \mathbf{F}_2 &= -\mathbf{R}_2 - m_2 g \mathbf{E}_y,\\ \mathbf{F} &= \mathbf{R}_1 - (m_1 + m_2)\, g \mathbf{E}_y.\end{aligned}$$

The resultant moments are

$$\begin{aligned}\mathbf{M}_{O_1} &= \bar{\mathbf{x}}_1 \times (2\mathbf{R}_2 - m_1 g \mathbf{E}_y) + \mathbf{M}_{c_1} + \mathbf{M}_{c_2},\\ \mathbf{M}_{O_2} &= -2\bar{\mathbf{x}}_1 \times \mathbf{R}_2 - \bar{\mathbf{x}}_2 \times m_2 g \mathbf{E}_y - \mathbf{M}_{c_2},\\ \mathbf{M}_O = \mathbf{M}_{O_1} + \mathbf{M}_{O_2} &= (\bar{\mathbf{x}}_1 \times -m_1 g \mathbf{E}_y) + (\bar{\mathbf{x}}_2 \times -m_2 g \mathbf{E}_y) + \mathbf{M}_{c_1}.\end{aligned}$$

10.2.3 Balance Laws and Analysis

Next, we examine conservation results. Clearly, $\mathbf{F} \neq \mathbf{0}$, even if gravity were absent, so $\mathbf{G}$ cannot be conserved. Next, we have that

$$\dot{\mathbf{H}}_O = \mathbf{M}_O = (\bar{\mathbf{x}}_1 \times -m_1 g \mathbf{E}_y) + (\bar{\mathbf{x}}_2 \times -m_2 g \mathbf{E}_y) + \mathbf{M}_{c_1}.$$

Because $\mathbf{M}_O$ has components in the $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ directions, no component of $\mathbf{H}_O$ is conserved. If gravity were absent, then $\dot{\mathbf{H}}_O = \mathbf{M}_O = \mathbf{M}_{c_1} = M_{c_{1_x}} \mathbf{E}_x + M_{c_{1_y}} \mathbf{E}_y$, and $\mathbf{H}_O \cdot \mathbf{E}_z$ would be conserved.

We now turn to the question of whether energy is conserved. To proceed, we start with the work-energy theorem for the system:

$$\frac{dT}{dt} = \mathbf{F}_1 \cdot \dot{\bar{\mathbf{x}}}_1 + \mathbf{M}_1 \cdot \boldsymbol{\omega}_1 + \mathbf{F}_2 \cdot \dot{\bar{\mathbf{x}}}_2 + \mathbf{M}_2 \cdot \boldsymbol{\omega}_2.$$

Substituting for the forces and moments listed above, we find with some rearranging that[5]

$$\begin{aligned}\frac{dT}{dt} &= \mathbf{R}_1 \cdot \mathbf{0} + \mathbf{R}_2 \cdot \mathbf{v}_A - m_1 g \mathbf{E}_y \cdot \bar{\mathbf{v}}_1 + (\mathbf{M}_{c_1} + \mathbf{M}_{c_2}) \cdot \boldsymbol{\omega}_1\\ &\quad -\mathbf{R}_2 \cdot \mathbf{v}_A - m_2 g \mathbf{E}_y \cdot \bar{\mathbf{v}}_2 - \mathbf{M}_{c_2} \cdot \boldsymbol{\omega}_2\\ &= -m_1 g \mathbf{E}_y \cdot \bar{\mathbf{v}}_1 - m_2 g \mathbf{E}_y \cdot \bar{\mathbf{v}}_2.\end{aligned}$$

Notice that we used the fact that the reaction moments are normal to the angular velocities, and, consequently, they do not contribute to the rate of change of kinetic energy. Manipulating the gravitational terms as usual and performing some obvious cancellations, we find that the rate of change of the total energy,

$$E = T + m_1 g \mathbf{E}_y \cdot \bar{\mathbf{x}}_1 + m_2 g \mathbf{E}_y \cdot \bar{\mathbf{x}}_2,$$

[5] Notice that after the rearranging, one has the alternative form of the work-energy theorem. For a single rigid body, this alternative form was discussed in Section 9.2.3 of Chapter 9.

is

$$\frac{dE}{dt} = 0.$$

In summary, if gravity is present, then only the total energy is conserved. On the other hand, if gravity were absent, then, in addition to energy conservation, $\mathbf{H}_O \cdot \mathbf{E}_z$ would also be conserved.

10.2.4 A Related Example

A related problem is to assume that the system is in motion. At some time $t = t_1$ the pin-joint at A freezes up so that $\dot{\theta}_1 = \dot{\theta}_2$ for $t > t_1$. Given $\theta_1(t_1^-)$, $\theta_2(t_1^-)$, $\dot{\theta}_1(t_1^-)$, and $\dot{\theta}_2(t_1^-)$, where t_1^- denotes the instant just before the freeze-up, is it possible to determine $\dot{\theta}_1(t_1^+) = \dot{\theta}_2(t_1^+) = \omega$, where t_1^+ denotes the instant just after the freeze-up?

The answer is yes! During the freeze-up, one can ignore the angular impulse due to gravity.[6] Then, from the integral form of the balance of angular momentum for the system, one has

$$\mathbf{H}_O(t_1^-) \cdot \mathbf{E}_z = \mathbf{H}_O(t_1^+) \cdot \mathbf{E}_z,$$

and this enables one to determine $\omega = \dot{\theta}_1(t_1^+) = \dot{\theta}_2(t_1^+)$. Explicitly,

$$\begin{aligned}
\mathbf{H}_O(t_1^-) \cdot \mathbf{E}_z = &\ \frac{m_1 L^2}{3}\dot{\theta}_1(t_1^-) + m_2\left(L^2\dot{\theta}_1(t_1^-) + \frac{3R^2}{2}\dot{\theta}_2(t_1^-)\right) \\
&+ m_2 RL\left(\dot{\theta}_2(t_1^-) + \dot{\theta}_1(t_1^-)\right)\cos\left(\theta_2(t_1^-) - \theta_1(t_1^-)\right), \\
\mathbf{H}_O(t_1^+) \cdot \mathbf{E}_z = &\ \left(\frac{m_1 L^2}{3} + m_2\left(L^2 + \frac{3R^2}{2}\right) + 2m_2 RL\cos\left(\theta_2(t_1^+) - \theta_1(t_1^+)\right)\right)\omega.
\end{aligned}$$

Equating these two expressions provides an equation to determine ω. One can also easily show that the energy is not conserved in this freeze-up, but we leave this as an exercise.

10.3 Impact of a Particle and a Rigid Body

We outline here some examples involving particles colliding with rigid bodies. As in collisions of particles with each other, one must be given some additional information: the coefficient of restitution for the problem at hand. Often, this is given implicitly. If the particle sticks to the body after the collision, one has that $e = 0$. If one is given the coefficient of restitution for these problems, it is important to note

[6] Essentially, one is assuming that the freeze-up occurs instantaneously.

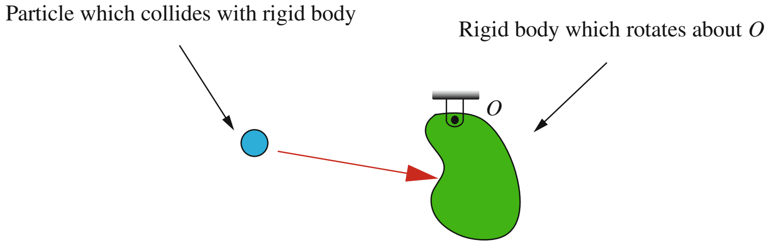

Fig. 10.5 *The first type of generic impact problem.*

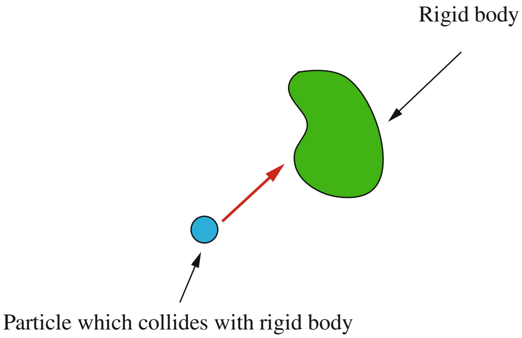

Fig. 10.6 *The second type of generic impact problem.*

that one of the velocity vectors pertains to the particle, whereas the other pertains to the velocity vector of the material point of the body that the particle impacts.

Our discussion here is brief, and we provide few analytical details. Indeed, what adds to the complexity of these problems is the calculation of linear and angular momenta before and after the impact. There are two types of problems common to undergraduate engineering courses in dynamics (see Figures 10.5 and 10.6). Normally, energy is not conserved in these problems.

The first type of problem is where the rigid body is attached by a pin-joint to a fixed point O (see Figure 10.5). For these problems, one assumes that the collision is instantaneous, and hence the angular momentum component $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved, where $\mathbf{H}_O$ is the angular momentum of the system relative to O. That is, one ignores any gravitational impulses during the collision. The linear momentum $\mathbf{G}$ of the system is not conserved because one has an impulse due to the reaction force at O. One then has a single equation: $\mathbf{H}_O \cdot \mathbf{E}_z$ is conserved. When the impact is such that the particle coalesces with the rigid body, this equation is usually used to determine the angular velocity of the particle-rigid body system immediately after the impact.

In the second type of problem, the rigid body is free to move on a plane or in space (see Figure 10.6). Again, one assumes that the collision is instantaneous, and hence the angular momentum component $\mathbf{H} \cdot \mathbf{E}_z$ is conserved. Here, $\mathbf{H}$ is the angular momentum of the system relative to the center of mass of the system. The linear momentum $\mathbf{G}$ of the system is also conserved for these problems. Hence, one has four conservations. When the impact is such that the particle coalesces with the rigid body, these four conservations are normally used to determine the velocity vector of the center of mass and angular velocity vector of the particle-rigid body system immediately after the impact.

10.4 Beyond This Primer

The scope of the present book was influenced by what is expected of a student in an undergraduate engineering dynamics course. What is beyond its scope are areas of active research. We now turn to two of these areas.

First, many readers will notice the emphasis on establishing differential equations for the motion of the system. The nature of the solutions to these equations contains many of the predictions of the model developed for the system. These solutions are therefore a crucial component of verifying any model. Not surprisingly, there has been an enormous amount of research on the solutions to differential equations that arise from the development of models for mechanical systems. Fueled by recent advances in numerical computations, this is still an active research area (see, e.g., Strogatz [76]). Based on our own experience, the predictions made by models of mechanical systems have usually been a source of enlightenment and improved understanding.

Another active research area is the development of more realistic models for impacting rigid bodies. These models are the basis for numerous simulations of vehicle collisions, and are also important in other applications. Of particular interest are models that incorporate frictional forces. The difficulties of establishing such theories was made evident in 1895 by Painlevé's paradoxical example of a rod sliding on a rigid horizontal surface [56].[7] Ruina [67] has also presented some interesting examples that illustrate some difficulties associated with the Coulomb friction laws we discussed in Chapter 4. We refer the interested reader to the review article by Stewart [74] for recent developments in this area.

10.5 Summary

This chapter has evident similarities to Chapter 7 where corresponding results for a system of particles were discussed. Rather than summarizing the main results presented in this chapter, it is probably more useful to give a verbal outline of how they were established.

First, for a system of particles and rigid bodies, the center of mass is calculated using the masses of the constituents, the position vectors of the particles, and the position vectors of the centers of mass of the rigid bodies. The linear momentum

[7] Paul Painlevé (1863–1933) was a French mathematician and politician. It is interesting to note that he is credited as being the first airplane passenger of Wilbur Wright in 1908.

$\mathbf{G}$ is calculated by summing the linear momenta of each of the constituents. This momentum is equal to the linear momentum of the center of mass of the system. Similarly, the angular momentum $\mathbf{H}_O$ is calculated by summing the angular momenta relative to O of the constituents. A similar remark applies for $\mathbf{H}$. Finally, the kinetic energy T of the system is calculated by summing the kinetic energies of the constituents. In general, $\mathbf{H}_O$ is not simply equal to the angular momentum of the center of mass of the system, and T is not simply equal to the kinetic energy of the center of mass of the system.

As in systems of particles, one can add the balance laws for the system of particles and rigid bodies to arrive at the balance laws

$$\mathbf{F} = \dot{\mathbf{G}}, \qquad \mathbf{M} = \dot{\mathbf{H}}, \qquad \mathbf{M}_O = \dot{\mathbf{H}}_O.$$

These laws are useful when establishing conservations of linear and angular momenta for the system of particles and rigid bodies. In addition, one can formulate integral forms of the balance laws:

$$\begin{aligned}
\mathbf{G}(t) - \mathbf{G}(t_0) &= \int_{t_0}^{t} \mathbf{F}(\tau)d\tau, \\
\mathbf{H}(t) - \mathbf{H}(t_0) &= \int_{t_0}^{t} \mathbf{M}(\tau)d\tau, \\
\mathbf{H}_O(t) - \mathbf{H}_O(t_0) &= \int_{t_0}^{t} \mathbf{M}_O(\tau)d\tau.
\end{aligned}$$

These results are very useful in impact problems.

Finally, the work-energy theorem for the system of particles and rigid bodies is obtained by adding the corresponding theorems for each of the constituents:

$$\frac{dT}{dt} = \sum_{k=1}^{K} {}_p\mathbf{F}_k \cdot \dot{\mathbf{r}}_k + \sum_{n=1}^{N} \left({}_r\mathbf{F}_n \cdot \dot{\bar{\mathbf{x}}}_n + \mathbf{M}_n \cdot \boldsymbol{\omega}_n\right).$$

As always, this theorem is useful for establishing conservation of energy results.

The main examples discussed in this chapter were a system of two rigid bodies and several impact problems. It is crucial to remember that obtaining the correct solutions to these problems depends on one's ability to establish expressions for $\mathbf{G}$, $\mathbf{H}_O$, and $\mathbf{H}$.

10.6 Exercises

The following short exercises are intended to assist you in reviewing the present chapter.

10.1. For the system discussed in Section 10.2, establish expressions for $\mathbf{H}_O$ when the disk of mass m_2 is replaced by a particle of mass m_2 that is attached at A. Is it possible to replace the system of the rigid rod and particle by a system consisting of a single rigid body?

10.2. For the system discussed in Exercise 10.1, derive an expression for the kinetic energy T.

10.3. Consider the system of rigid bodies discussed in Section 10.2. How do the results for $\mathbf{H}_O$ and T simplify if the disk were pin-jointed at its center of mass to A?

10.4. For the system discussed in Section 10.2, derive expressions for $\mathbf{H}_O$ and T if the disk were welded at its center of mass to A.

10.5. For the system of two rigid bodies discussed in Section 10.2, derive expressions for $\mathbf{H}_O$ and T if the disk were replaced by a rigid rod of length $2R$.

10.6. A circular disk of mass m_1 and radius R lies at rest on a horizontal plane. The origin of the coordinate system is taken to coincide with the center of mass of the disk. At an instant in time t_1, a particle of mass m_2 which has a velocity vector $\mathbf{v} = v_x\mathbf{E}_x + v_y\mathbf{E}_y$ collides with the disk. The collision occurs at the point of the disk whose position vector is $R\cos(\phi)\mathbf{E}_x + R\sin(\phi)\mathbf{E}_y$. After the impact, the particle adheres to the disk. Show that the position vector of the center of mass of the system during the instant of impact is

$$\bar{\mathbf{x}} = \frac{m_2 R}{m_1 + m_2}\left(\cos(\phi)\mathbf{E}_x + \sin(\phi)\mathbf{E}_y\right).$$

In addition, show that the velocity vector of the center of mass of the system immediately following the impact is

$$\bar{\mathbf{v}}\left(t_1^+\right) = \frac{m_2}{m_1 + m_2}\left(v_x\mathbf{E}_x + v_y\mathbf{E}_y\right).$$

10.7. For the system discussed in Exercise 10.6, show that the angular momentum of the system relative to its center of mass at the instant prior to the collision is

$$\mathbf{H}(t_1^-) = \frac{m_1 m_2 R}{m_1 + m_2}\left(v_y\cos(\phi) - v_x\sin(\phi)\right)\mathbf{E}_z.$$

In addition, show that the angular momentum of the system relative to its center of mass immediately after the collision is

$$\mathbf{H}\left(t_1^+\right) = \left(\frac{m_2}{m_1 + m_2} + \frac{1}{2}\right) m_1 R^2 \omega\left(t_1^+\right)\mathbf{E}_z,$$

where $\omega\left(t_1^+\right)\mathbf{E}_z$ is the angular velocity vector of the system immediately after the collision.

10.8. Using the results of Exercise 10.7, determine the angular velocity vector of the system immediately following the impact discussed in Exercise 10.6. Under which conditions is it possible for this velocity vector to be $\mathbf{0}$?

10.9. Determine the kinetic energy lost during the collision discussed in Exercise 10.6.

10.10. Which modifications to the results of Exercises 10.6 to 10.8 are required to accommodate the situation where the center of mass of the disk was in motion at the instant prior to impact?

10.11. Repeat Exercises 10.6 through 10.9 for the case where the disk is pinned at its center to a fixed point O.

Part V
Appendices

Appendix A
Preliminaries on Vectors and Calculus

CAVEAT LECTOR

In writing this primer, I have assumed that the reader has had courses in linear algebra and calculus. This being so, I have more often than not found that these topics have been forgotten. Here, I review some of the basics. But it is a terse review, and I strongly recommend that readers review their own class notes and other texts on these topics in order to fill the gaps in their knowledge.

Students who are able to differentiate vectors and are familiar with the chain and product rules of calculus have a distinct advantage in comprehending the material in this primer and in other courses. I have never been able to sufficiently emphasize this point to students at the beginning of an undergraduate dynamics course.

A.1 Vector Notation

A fixed (right-handed) Cartesian basis for Euclidean three-space $\mathbb{E}^3$ is denoted by the set $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$. These three vectors are orthonormal (i.e., they each have a unit magnitude and are mutually perpendicular).

For any vector $\mathbf{b}$, one has the representation

$$\mathbf{b} = b_x\mathbf{E}_x + b_y\mathbf{E}_y + b_z\mathbf{E}_z,$$

where b_x, b_y, and b_z are the Cartesian components of the vector $\mathbf{b}$ (cf. Figure A.1).

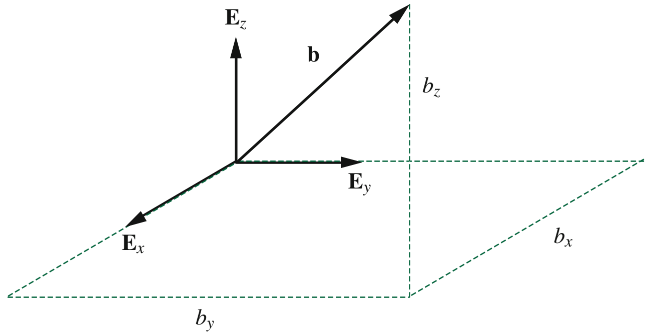

Fig. A.1 *A vector* $\mathbf{b}$ *and its three components relative to the basis* $\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$.

A.2 The Dot Product and The Cross Product

The two most commonly used vector products are the dot and cross products. The dot product of any two vectors $\mathbf{u}$ and $\mathbf{w}$ in $\mathbb{E}^3$ is a scalar defined by

$$\mathbf{u} \cdot \mathbf{w} = u_x w_x + u_y w_y + u_z w_z = \|\mathbf{u}\| \, \|\mathbf{w}\| \cos(\gamma),$$

where γ is the angle subtended by $\mathbf{u}$ and $\mathbf{w}$, and $\|\mathbf{b}\|$ denotes the norm (or magnitude) of a vector $\mathbf{b}$:

$$\|\mathbf{b}\|^2 = \mathbf{b} \cdot \mathbf{b} = b_x^2 + b_y^2 + b_z^2.$$

Clearly, if two vectors are perpendicular to each other, then their dot product is zero.

One can use the dot product to define a unit vector $\mathbf{n}$ in the direction of any vector $\mathbf{b}$:

$$\mathbf{n} = \frac{\mathbf{b}}{\|\mathbf{b}\|}.$$

This formula is very useful in establishing expressions for friction forces and normal forces.

The cross product of any two vectors $\mathbf{b}$ and $\mathbf{c}$ is defined as

$$\mathbf{b} \times \mathbf{c} = (b_y c_z - b_z c_y)\,\mathbf{E}_x + (b_z c_x - b_x c_z)\,\mathbf{E}_y + (b_x c_y - b_y c_x)\,\mathbf{E}_z.$$

This expression for the cross product can be expressed in another form involving the determinant of a matrix:

$$\mathbf{b} \times \mathbf{c} = -\mathbf{c} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix}.$$

It follows from the definition of the cross product that $\mathbf{b} \times \mathbf{c}$ is a vector that is normal to the plane formed by $\mathbf{b}$ and $\mathbf{c}$. You should also notice that if two vectors are parallel, then their cross product is the zero vector $\mathbf{0}$.

A.3 Differentiation of Vectors

Given a vector $\mathbf{u}$, suppose it is a function of time t: $\mathbf{u} = \mathbf{u}(t)$. One can evaluate its derivative using the product rule:

$$\frac{d\mathbf{u}}{dt} = \frac{du_x}{dt}\mathbf{E}_x + \frac{du_y}{dt}\mathbf{E}_y + \frac{du_z}{dt}\mathbf{E}_z + u_x\frac{d\mathbf{E}_x}{dt} + u_y\frac{d\mathbf{E}_y}{dt} + u_z\frac{d\mathbf{E}_z}{dt}.$$

However, $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ are constant vectors (i.e., they have constant magnitude and direction). Hence, their time derivatives are zero, and the expression for the time derivative of $\mathbf{u}$ simplifies to

$$\frac{d\mathbf{u}}{dt} = \frac{du_x}{dt}\mathbf{E}_x + \frac{du_y}{dt}\mathbf{E}_y + \frac{du_z}{dt}\mathbf{E}_z.$$

We can also use the product rule of calculus to show that

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{w}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{w} + \mathbf{u}\cdot\frac{d\mathbf{w}}{dt}, \qquad \frac{d}{dt}(\mathbf{u}\times\mathbf{w}) = \frac{d\mathbf{u}}{dt}\times\mathbf{w} + \mathbf{u}\times\frac{d\mathbf{w}}{dt}.$$

These results are obtained by representing the vectors $\mathbf{u}$ and $\mathbf{w}$ with respect to a Cartesian basis, then evaluating the left- and right-hand sides of both equations and showing their equality.

To differentiate any vector-valued function $\mathbf{c}(s(t))$ with respect to t, we use the chain rule:

$$\frac{d\mathbf{c}}{dt} = \frac{ds}{dt}\frac{d\mathbf{c}}{ds} = \frac{ds}{dt}\left(\frac{dc_x}{ds}\mathbf{E}_x + \frac{dc_y}{ds}\mathbf{E}_y + \frac{dc_z}{ds}\mathbf{E}_z\right).$$

For example, suppose $s = t^2$ and $\mathbf{c} = s^2\mathbf{E}_z$. Then, $d\mathbf{c}/dt = 4t^3\mathbf{E}_z$.

A.4 A Ubiquitous Example of Vector Differentiation

One of the main sets of vectors arising in any course on dynamics is $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$:

$$\begin{aligned}
\mathbf{e}_r &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y,\\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y,\\
\mathbf{e}_z &= \mathbf{E}_z.
\end{aligned}$$

We also refer the reader to Figure A.2. In the above equations, θ is a function of time.

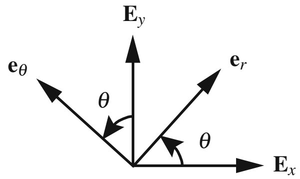

Fig. A.2 *The unit vectors* $\mathbf{e}_r$ *and* $\mathbf{e}_\theta$.

Using the previous developments, you should be able to establish that

$$\begin{aligned}
\frac{d\mathbf{e}_r}{d\theta} &= -\sin(\theta)\mathbf{E}_x + \cos(\theta)\mathbf{E}_y = \mathbf{e}_\theta, \\
\frac{d\mathbf{e}_\theta}{d\theta} &= -\cos(\theta)\mathbf{E}_x - \sin(\theta)\mathbf{E}_y = -\mathbf{e}_r, \\
\frac{d\mathbf{e}_r}{dt} &= \frac{d\theta}{dt}\mathbf{e}_\theta, \\
\frac{d\mathbf{e}_\theta}{dt} &= -\frac{d\theta}{dt}\mathbf{e}_r.
\end{aligned}$$

A useful exercise is to evaluate these expressions and graphically represent them for a given $\theta(t)$. For example, $\theta(t) = 10t^2 + 15t$.

Finally, you should be able to show that

$$\begin{array}{ccc}
\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_z, & \mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\theta, & \mathbf{e}_\theta \times \mathbf{e}_z = \mathbf{e}_r, \\
\mathbf{e}_r \cdot \mathbf{e}_r = 1, & \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1, & \mathbf{e}_z \cdot \mathbf{e}_z = 1, \\
\mathbf{e}_r \cdot \mathbf{e}_\theta = 0, & \mathbf{e}_\theta \cdot \mathbf{e}_z = 0, & \mathbf{e}_r \cdot \mathbf{e}_z = 0.
\end{array}$$

In other words, $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ forms an orthonormal set of vectors. Furthermore, because $\mathbf{e}_z \cdot (\mathbf{e}_r \times \mathbf{e}_\theta) = 1$, this set of vectors is also right-handed.

A.5 Ordinary Differential Equations

The main types of differential equations appearing in undergraduate dynamics courses are of the form $\ddot{u} = f(u)$, where the superposed double dot indicates the second derivative of u with respect to t. The general solution of this differential equation involves two constants: the initial conditions for $u(t_0) = u_0$ and its velocity $\dot{u}(t_0) = \dot{u}_0$. Often, one chooses time such that $t_0 = 0$.

The most comprehensive source of mechanics problems that involve differential equations of the form $\ddot{u} = f(u)$ is Whittaker's classical work [81]. It should also be added that classical works in dynamics placed tremendous emphasis on obtaining analytical solutions to such equations. Recently, the engineering dynamics community has become increasingly aware of possible chaotic solutions. Consequently, the existence of analytical solutions is generally not anticipated. We refer the reader to

Moon [49] and Strogatz [76] for further discussions on, and references to, this matter. Further perspectives can be gained by reading the books by Barrow-Green [3] and Diacu and Holmes [22] on Henri Poincaré's seminal work on chaos, and Peterson's book [61] on chaos in the solar system.

For many of the examples discussed in this book, the differential equations of motion were solved numerically. These examples include the simple pendulum's motion shown in Figure 2.1 in Chapter 2 and the motions of the whirling particles shown in Figure 7.5 in Chapter 7. There are several well-established computation packages, such as MATLAB and MATHEMATICA, which are capable of performing the numerical integrations and generating graphics of the resulting solutions. For further details on how these respective programs can be used, we recommend the texts of Palm [57] and Hunt et al. [38].

A.5.1 The Projectile Problem

Arguably the easiest set of differential equations in this book makes its appearance when studying the motion of a particle under the influence of a gravitational force $-mg\mathbf{E}_y$. As shown in Section 1.6 of Chapter 1, the differential equations governing the motion of the particle are

$$m\ddot{x} = 0, \qquad m\ddot{y} = -mg, \qquad m\ddot{z} = 0.$$

Clearly, each of these three equations is of the form $\ddot{u} = f(u)$. The general solution to the second of these equations is

$$y(t) = y_0 + \dot{y}_0 (t - t_0) - \frac{g}{2} (t - t_0)^2 .$$

Here, $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$ are the initial conditions. You should verify the solution for $y(t)$ given above by first examining whether it satisfies the initial conditions and then seeing whether it satisfies the differential equation $\ddot{y} = -g$. By setting $g = 0$ and changing variables from y to x and z, the solutions to the other two differential equations can be obtained.

A.5.2 The Harmonic Oscillator

The most common example of a differential equation in mechanical engineering is found from the harmonic oscillator. Here, a particle of mass m is attached by a linear spring of stiffness K to a fixed point. The variable x is chosen to measure both the displacement of the particle and the displacement of the spring from its unstretched state. The governing differential equation is

$$m\ddot{x} = -Kx.$$

This equation has the general solution

$$x(t) = x_0 \cos\left(\omega_0 \left(t - t_0\right)\right) + \frac{\dot{x}_0}{\omega_0} \sin\left(\omega_0 \left(t - t_0\right)\right),$$

where $x\left(t_0\right) = x_0$ and $\dot{x}\left(t_0\right) = \dot{x}_0$ are the initial conditions, and $\omega_0 = \sqrt{K/m}$ is the natural frequency of the oscillator.

A.5.3 A Particle in a Whirling Tube

The penultimate example of interest arises in problems featuring a particle of mass m that is in motion in a smooth frictionless tube. The tube is being rotated in a horizontal plane with a constant angular speed Ω. The differential equation governing the radial motion of the particle is

$$m\ddot{r} = m\Omega^2 r.$$

This equation has the general solution

$$r(t) = r_0 \cosh\left(\Omega \left(t - t_0\right)\right) + \frac{\dot{r}_0}{\Omega} \sinh\left(\Omega \left(t - t_0\right)\right),$$

where $r\left(t_0\right) = r_0$ and $\dot{r}\left(t_0\right) = \dot{r}_0$ are the initial conditions.

A.5.4 The Planar Pendulum

Our final example of a differential equation of the form $\ddot{u} = f(u)$ arises in the planar pendulum discussed in Section 2.4 of Chapter 2. Recall that the equation governing the motion of the pendulum was

$$mL\ddot{\theta} = -mg\cos\left(\theta\right).$$

This equation is of the form discussed above with $u = \theta$ and $f(u) = -g\cos(u)/L$. Here, f is a nonlinear function of u. Given the initial conditions $\theta\left(t_0\right) = \theta_0$ and $\dot{\theta}\left(t_0\right) = \dot{\theta}_0$, this differential equation can be solved analytically. The resulting solution involves special functions that are known as Jacobi's elliptic functions.[1] An alternative method of solution is to use numerical integration. For instance, the func-

[1] A discussion of these functions, in addition to the analytical solution of the particle's motion, can be found in Lawden [43], for instance.

tion NDSOLVE in MATHEMATICA was used to generate the pendulum motions shown in Figure 2.1. It was also used to solve the equations of motion of the hanging rod shown in Figure 9.7.

Elliptic functions are beyond the scope of an undergraduate dynamics class, so instead one is normally asked to use the conservation of the total energy E of the particle to solve most posed problems involving this pendulum.

Appendix B
Weekly Course Content and Notation in Other Texts

ABBREVIATIONS

For convenience in this appendix we use the following abbreviations: BF, Bedford and Fowler [6]; BJ, Beer and Johnston [7]; H, Hibbeler [36]; MK, Meriam and Kraige [48]; RS, Riley and Sturges [63]; and S, Shames [69].

B.1 Weekly Course Content

The following is an outline for a 15-week (semester-long) course in undergraduate engineering dynamics. Here, we list the weekly topics along with the corresponding sections in this primer. We also indicate the corresponding sections in other texts. This correspondence is, of course, approximate: all of the cited texts have differences in scope and emphasis.

Normally, the course is divided into three parts: a single particle, systems of particles, and (planar dynamics of) rigid bodies. The developments in most texts also cover the material in this order, the exception being Riley and Sturges [63].

Week	Topic	Primer Section	Other Texts
1	Single Particle: Cartesian Coordinates	Chapter 1	BF: Ch.1, 2.1–2.3, 3.1–3.4 BJ: 11.1–11.11, 12.5 H: 12.1–12.6, 13.4 MK: 1/1–1/7, 2/2, 2/4, 3/4 RS: 13.1–13.4, 15.1–15.3 S: 11.1–11.4, 12.1–12.4
2	Single Particle: Polar Coordinates	Chapter 2	BF: 2.3, 3.4 BJ: 11.14, 12.8 H: 12.8, 13.6 MK: 2/6, 3/5 RS: 13.5, 13.7, 15.4 S: 11.6, 12.5
3	Single Particle: Serret-Frenet Triads	Chapter 3	BF: 2.3, 3.4 BJ: 11.13, 12.5 H: 12.7, 13.5 MK: 2/5, 2/7, 3/5 RS: 13.5, 13.7, 15.4 S: 11.5, 12.9
4	Single Particle: Further Kinetics	Chapter 4	BF: 3.4 BJ: 12.5 H: 13.4–13.6 MK: 3/5 RS: 15.3, 15.4 S: 12.4, 12.5, 12.9
5	Single Particle: Work and Energy	Chapter 5	BF: Ch. 4 BJ: 13.1–13.9 H: 14.1, 14.2, 14.4–14.6 MK: 3/6, 3/7 RS: 17.1–17.3, 17.5–17.10 S: 13.1–13.5
6	Linear and Angular Momenta	Chapter 6 Sects. 1 & 2	BF: 5.1, 5.2, 5.4 BJ: 12.2, 12.7, 12.9,13.11 H: 15.1, 15.5–15.7 MK: 3/9, 3/10 RS: 19.2, 19.5 S: 14.1, 14.3, 14.6

Week	Topic	Primer Section	Other Texts
7	Collisions of Particles	Chapter 6 Sects. 3–5	BF: 5.3 BJ: 13.12–13.14 H: 14.3, 14.6, 15.4 MK: 3/12 RS: 19.4 S: 14.4–14.5
8	Systems of Particles	Chapter 7	BF: 7.1, 8.1 BJ: 14.1–14.9 H: 13.3, 14.3, 14.6, 15.3 MK: 4/1–4/5 RS: 17.4–17.8, 19.3, 19.5 S: 12.10, 14.2, 14.7, 13.6–13.9
9	Kinematics of Rigid Bodies	Chapter 8	BF: 6.1–6.3 BJ: 15.1–15.4 H: 16.1–16.4 MK: 5/1–5/4 RS: 14.1–14.3 S: 15.1–15.5
10	Kinematics of Rigid Bodies	Chapter 8	BF: 6.4–6.6 BJ: 15.4–15.8, 15.10–15.15 H: 16.4–16.8 MK: 5/5–5/7 RS: 14.4–14.6 S: 15.5–15.11
11	Planar Dynamics of Rigid Bodies	Chapter 9	BF: 7.2–7.3, App., 9.2 BJ: Ch. 16 H: 21.1, 21.2, 17.3 MK: 6/1–6/3, Apps. A & B RS: 16.2, 16.3, 20.6 S: 16.1–16.4
12	Planar Dynamics of Rigid Bodies	Chapter 9	BF: 7.4 BJ: Ch. 16 H: 17.4 MK: 6/4 RS: 16.4 S: 16.5

Week	Topic	Primer Section	Other Texts
13	Planar Dynamics of Rigid Bodies	Chapter 9	BF: 8.1–8.3 BJ: Ch. 16 & 17.1–17.7 H: 17.5 & Ch. 18 MK: 6/5. 6/6 RS: 16.4 & Ch. 18 S: 16.6, 17.1–17.3
14	Planar Dynamics of Rigid Bodies	Chapter 10	BF: 8.4 BJ: 17.8–17.11 H: Ch. 19 MK: 6/8 RS: 20.1–20.5 S: 17.4–17.7
15	Vibrations	Not Covered	BF: Ch. 10 BJ: Ch. 19 H: Ch. 22 MK: Ch. 8 RS: Ch. 21 S: Ch. 22

B.2 Notation in Other Texts

Here, we give a brief summary of some of the notational differences between this primer and those used in other texts. In many of the cited texts only plane curves are considered. Consequently, the binormal vector $\mathbf{e}_b$ is not explicitly mentioned.

	Primer Notation	**Other Texts**
Cartesian Basis Vectors	$\{\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z\}$	BF: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		BJ: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		H: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		MK: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		RS: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		S: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
Serret-Frenet Triad	$\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$	BF: $\{\mathbf{e}_t, \mathbf{e}_n, -\}$
		BJ: $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$
		H: $\{\mathbf{u}_t, \mathbf{u}_n, \mathbf{u}_b\}$
		MK: $\{\mathbf{e}_t, \mathbf{e}_n, -\}$
		RS: $\{\mathbf{e}_t, \mathbf{e}_n, -\}$
		S: $\{\varepsilon_t, \varepsilon_n, \varepsilon_t \times \varepsilon_n\}$
Linear Momentum of a Particle	$\mathbf{G} = m\mathbf{v}$	BF: $m\mathbf{v}$
		BJ: $\mathbf{L} = m\mathbf{v}$
		H: $m\mathbf{v}$
		MK: $\mathbf{G}$
		RS: $\mathbf{L} = m\mathbf{v}$
		S: $m\mathbf{V}$
Corotational Basis or Body Fixed Basis	$\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$	BF: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		BJ: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		H: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		MK: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
		RS: $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$
		S: $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$

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[9] The title of this paper translates to "On several formulae relating to curves of double curvature." According to Spivak [73], curves of double curvature is an old term for space curves.

Index