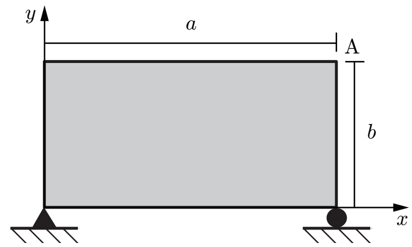
Problem #1



A $a \times b$ rectangular plate of thickness t made of an isotropic linear elastic material with Lamé constants λ and μ is subjected to the in-plane strains

$$\varepsilon_{xx} = \left(\frac{x}{a} + \frac{xy}{ab} + C\,\frac{y^2}{b^2} \right) \cdot 10^{-4} \,, \qquad \varepsilon_{yy} = \left(-\frac{xy}{ab} \right) \cdot 10^{-4} \,, \qquad \varepsilon_{xy} = \left(\frac{y}{b} - \frac{x}{a} \right) \cdot 10^{-4} \,,$$

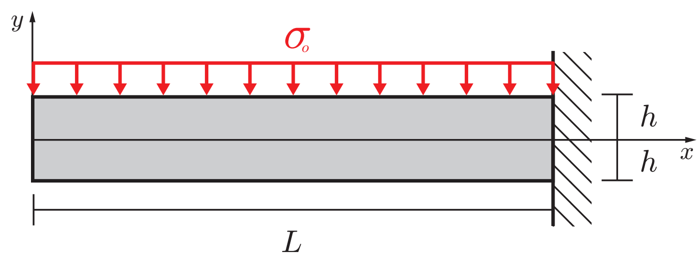
for an undetermined constant C, while kept under plane strain conditions in the thickness direction. Determine:

1. The value of C so the strain field is compatible (assume that value in all items below).
2. The displacement of the plate's top right corner (Point A).
3. The minimum uniaxial yield limit σ_{yp} of the material with a factor of safety of 1.5 so no yielding is observed at Point A according to von Mises yield criterion.

Problem #2

A beam of length L and height $2h$ is subjected to a uniform normal stress σ_o on its top while being fixed at its right end. The left end is free of any loading in resultant form. There is no body force. If the beam's stresses are to be represented by the Airy's stress function

$$\phi(x,y) = C_1\,x^2 + C_2\,x^2 y + C_3\,y^3 + C_4\,y^5 + C_5\,x^2 y^3$$

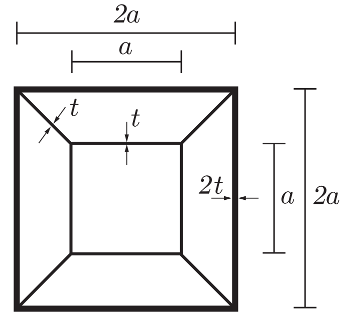


for five constant C_i, determine:

1. The constants C_i.
2. The stresses at the right support. Sketch their distribution, indicating the values at the top and bottom. How does this solution compare with classical beam theory?
3. Obtain the reacting force and moment at the right support by direct integration of the stresses in Item 2. Is global equilibrium satisfied?

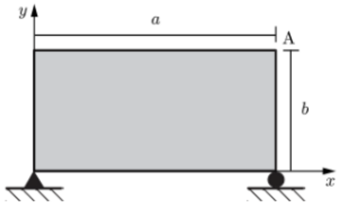
Problem #3

A shaft has the multi-cell cross section shown in the figure, with an outer $2a \times 2a$ square cell of thickness $2t$ and an inner $a \times a$ square cell of thickness t, both cells connected by walls of thickness t (with $t \ll a$). Determine:



1. The torsional constant J.
2. The maximum elastic torque T_{yp} for a yield limit in shear τ_{yp}.
3. How much more torque percentage-wise over T_{yp} can be applied to the section if the material can take τ_{yp} at most in shear?

Problem #1



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for an undetermined constant C, while kept under plane strain conditions in the thickness direction. Determine:

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Work with strains $\varepsilon \cdot 10^{-4}$ (i.e. up to factor 10^{-4})

① Compatibility (plane strain)

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 2\frac{C}{b^2} + 0 - 0 = 0 \Rightarrow \boxed{C = 0}$$

② Displacement field

$$\frac{\partial u}{\partial x} = \varepsilon_{xx} = \frac{x}{a} + \frac{xy}{ab} \Rightarrow u = \frac{x^2}{2a} + \frac{x^2 y}{2ab} + f(y)$$

$$\frac{\partial v}{\partial y} = \varepsilon_{yy} = -\frac{xy}{ab} \Rightarrow v = -\frac{xy^2}{2ab} + g(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{x^2}{2ab} + f' - \frac{y^2}{2ab} + g' = 2\varepsilon_{xy} = 2\left(\frac{y}{b} - \frac{x}{a}\right)$$

$$\Rightarrow \underbrace{\left(\frac{x^2}{2ab} + \frac{2x}{a} + g'\right)}_{+A} + \underbrace{\left(-\frac{y^2}{2ab} - \frac{2y}{b} + f'\right)}_{-A} = 0$$

$$\Rightarrow g(x) = -\frac{x^3}{6ab} - \frac{x^2}{a} + Ax + B \qquad f(y) = \frac{y^3}{6ab} + \frac{y^2}{b} - Ay + C$$

Boundary conditions: $u(0,0) = v(0,0) = 0 \Rightarrow B = C = 0$

$v(a,0) = 0 \Rightarrow g(a) = -\frac{a^2}{6b} - a + Aa = 0 \Rightarrow A = 1 + \frac{a}{6b}$

At Point A (a,b)

$$\boxed{\begin{aligned} u_A &= \left(\frac{5a}{6} + \frac{b^2}{6a}\right) \cdot 10^{-4} \\ v_A &= \left(-\frac{b}{2}\right) \cdot 10^{-4} \end{aligned}}$$

③ Stresses $\quad \underline{\sigma} = \lambda \, \mathrm{tr}\, \underline{\varepsilon}\, \underline{\mathbb{1}} + 2\mu\, \underline{\varepsilon}$

$$\mathrm{tr}\,\underline{\varepsilon} = \varepsilon_{xx} + \varepsilon_{yy} + \overset{=0}{\varepsilon_{zz}} = \frac{x}{a}\cdot 10^{-4}$$

$\cdot 10^{-4}$

$$\Rightarrow \sigma_{xx} = \lambda\,\mathrm{tr}\,\underline{\varepsilon} + 2\mu\,\varepsilon_{xx} = \lambda\frac{x}{a} + 2\mu\left(\frac{x}{a} + \frac{xy}{ab}\right)$$

$$\sigma_{yy} = \lambda\,\mathrm{tr}\,\underline{\varepsilon} + 2\mu\,\varepsilon_{yy} = \lambda\frac{x}{a} - 2\mu\frac{xy}{ab}$$

$$\sigma_{zz} = \lambda\,\mathrm{tr}\,\underline{\varepsilon} + 2\mu\,\varepsilon_{zz} = \lambda\frac{x}{a}$$

$$\sigma_{xy} = 2\mu\,\varepsilon_{xy} = 2\mu\left(\frac{y}{b} - \frac{x}{a}\right)$$

At Point $A = (a, b)$

$$\Rightarrow \sigma_{xx} = (\lambda + 4\mu)\,10^{-4} = \sigma_1$$

$$\sigma_{yy} = (\lambda - 2\mu)\,10^{-4} = \sigma_2$$

$$\sigma_{zz} = \lambda\cdot 10^{-4} = \sigma_3$$

principal stresses

$$\sigma_{xy} = 0 \quad \Rightarrow \text{principal directions}$$

von Mises

$$2\sigma_{yp}^2 \geq (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2$$

$$= \left[(6\mu)^2 + (4\mu)^2 + (2\mu)^2\right]10^{-8} = 56\mu^2\,10^{-8}$$

$$\Rightarrow \sigma_{yp} \geq 2\sqrt{7}\,\mu\,10^{-4}$$

$\Rightarrow$ Minimum σ_{yp} with $(FS = 1.50) = 1.5\cdot 2\sqrt{7}\,\mu\,10^{-4}$

$$\Rightarrow \boxed{\sigma_{yp} \geq 3\sqrt{7}\,\mu\cdot 10^{-4}}$$

Problem #2

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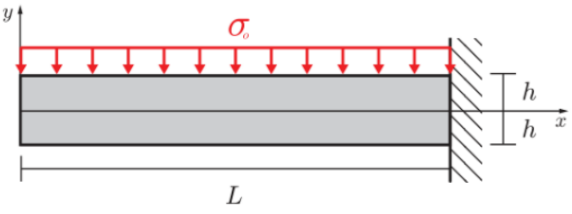

$$\phi(x,y) = C_1\,x^2 + C_2\,x^2y + C_3\,y^3 + C_4\,y^5 + C_5\,x^2y^3$$

for five constant C_i, determine:

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① Check compatibility: $\nabla^4\phi = 0$

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^2\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 + 2C_5\,2\cdot3\cdot2y + C_4\,5\cdot4\cdot3\cdot2\,y = 0$$

$$\Rightarrow C_5 + 5C_4 = 0 \Rightarrow C_5 = -5C_4$$

$$\Rightarrow \phi(x,y) = C_1x^2 + C_2x^2y + C_3y^3 + C_4\left(y^5 - 5x^2y^3\right)$$

Stresses:

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 6C_3y + 10C_4\left(2y^3 - 3x^2y\right)$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 2C_1 + 2C_2y - 10C_4y^3$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -2C_2x + 30C_4xy^2$$

Boundary conditions:

$$0 = \sigma_{xy}\Big|_{y=\pm h} = 2x\left(-C_2 + 15C_4h^2\right) \Rightarrow C_2 = 15C_4h^2$$

$$\left.\begin{aligned} 0 &= \sigma_{yy}\big|_{y=-h} = 2C_1 - 20C_4h^3 \\ -\sigma_0 &= \sigma_{yy}\big|_{y=+h} = 2C_1 + 20C_4h^3 \end{aligned}\right\} \Rightarrow \quad C_1 = -\frac{\sigma_0}{4}, \quad C_4 = -\frac{\sigma_0}{40h^3}$$

$$\Rightarrow C_2 = -\frac{3}{8}\frac{\sigma_0}{h}$$

At the left end: zero force, zero moment

Note that $\sigma_{xy}=0$ at $x=0$ ✓

$$0=\int_{-h}^{h}\sigma_{xx}\Big|_{x=0} t\,dy = t\int_{-h}^{h}\left[6C_3 y + 20C_4 y^3\right]dy = 0 \quad ✓$$

$$0=\int_{-h}^{h} y\,\sigma_{xx}\Big|_{x=0} t\,dy = t\int_{-h}^{h}\left[6C_3 y^2 + 20C_4 y^4\right]dy = 4C_3h^3t + 8C_4h^5t$$

$$\Rightarrow C_3 = -2C_4h^2 = \frac{\sigma_0}{20h}$$

$$\boxed{C_1 = -\frac{\sigma_0}{4} \qquad C_2 = -\frac{3}{8}\frac{\sigma_0}{h} \qquad C_3 = \frac{\sigma_0}{20h} \qquad C_4 = -\frac{\sigma_0}{40h^3} \qquad C_5 = +\frac{\sigma_0}{8h^3}}$$

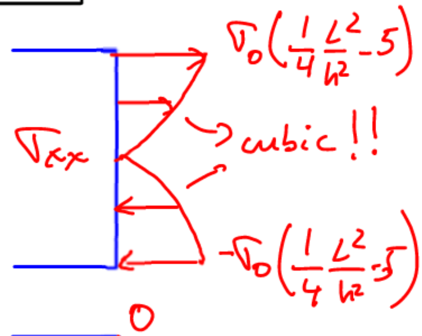
② At $x=L$

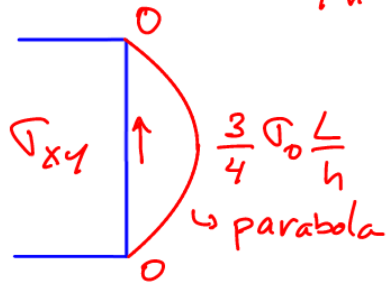
$$\sigma_{xx} = \frac{\sigma_0}{20}\left[\left(6+15\frac{L^2}{h^2}\right)\frac{y}{h} - 10\frac{y^3}{h^3}\right]$$

$$= \underbrace{\frac{\sigma_0}{20}\left[6\frac{y}{h} - 10\frac{y^3}{h^3}\right]}_{\text{cubic correction}} + \underbrace{\frac{\frac{1}{2}\sigma_0L^2}{\frac{1}{12}8h^3}y}_{\frac{M}{I}y}$$

classical beam theory

$$\sigma_{xy} = \frac{3}{4}\sigma_0\frac{L}{h}\left(1-\frac{y^2}{h^2}\right)$$

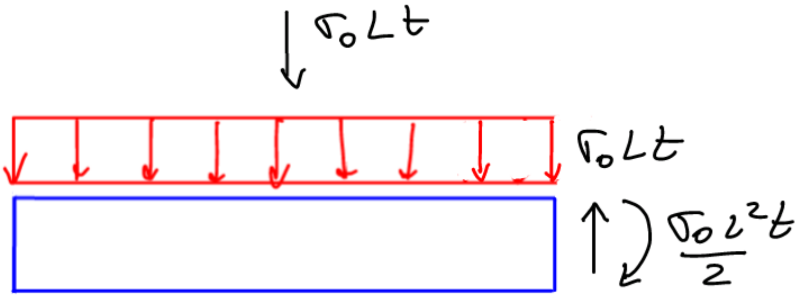




③ Reactions

$$F_x = \int_{-h}^{h}\sigma_{xx}\,t\,dy = 0 \qquad F_y = \int_{-h}^{h}\sigma_{xy}\,t\,dy = \frac{3}{4}\sigma_0\frac{L}{h}\left(2h - \frac{2h^3}{3h^2}\right) = \sigma_0 L t$$

$$M = \int_{-h}^{h} y\,\sigma_{xx}\,t\,dy = \frac{\sigma_0}{20}\left[\left(6+15\frac{L^2}{h^2}\right)\frac{2}{3}h^2 - \frac{10}{5}2h^2\right]t = \frac{\sigma_0}{2}L^2 t$$

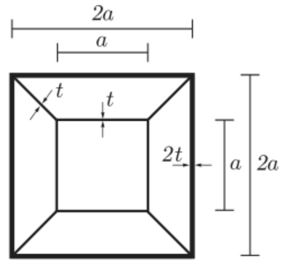


$$\Sigma F = \sigma_0 L L - \sigma_0 L t = 0 \quad ✓$$

$$\Sigma M = \sigma_0 L t\frac{L}{2} - \frac{\sigma_0 L^2}{2}t = 0 \quad ✓$$

Global equilibrium is satisfied

Problem #3



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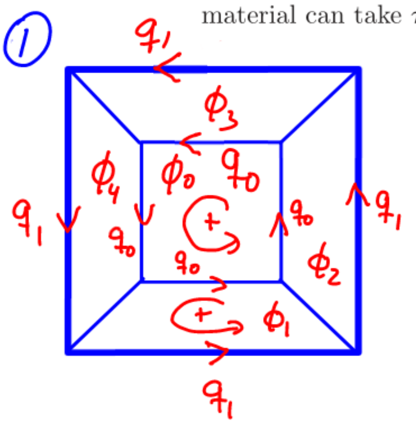
①



By symmetry $\phi_1 = \phi_2 = \phi_3 = \phi_4$

$\Rightarrow q = 0$ along inclined walls

$q_1 = \phi_1 \qquad q_0 = \phi_0 - \phi_1$

$$\oint_{C_1} \tau\, ds = \oint_{C_1} \frac{q(s)}{t(s)}\, ds = \frac{\phi_1}{2t} 2a - \frac{\phi_0 - \phi_1}{t} a = 2G\Theta \overbrace{\frac{1}{4}(4a^2 - a^2)}^{= A_1}$$

$$\oint_{C_0} \tau\, ds = \oint_{C_0} \frac{q(s)}{t(s)}\, ds = \frac{\phi_0 - \phi_1}{t} 4a = 2G\Theta \overbrace{a^2}^{= A_0}$$

$$\Rightarrow \left.\begin{aligned} 2\phi_1 - \phi_0 &= \frac{3}{2} a t G\Theta \\ -\phi_1 + \phi_0 &= \frac{1}{2} a t G\Theta \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \phi_1 &= 2atG\Theta \\ \phi_0 &= \frac{5}{2} atG\Theta \end{aligned}}$$

$$\Rightarrow T = 2\left[4A_1\phi_1 + \phi_0 A_0\right] = 2\left[3\phi_1 + \phi_0\right] a^2 = 17 a^3 t G\Theta$$

$$\Theta = \frac{T}{GJ} \Rightarrow J = \frac{T}{G\Theta} = 17a^3 t \qquad \boxed{J = 17a^3 t}$$

② Stresses:

$$\tau_0 = \frac{q_0}{t} = \frac{\phi_0 - \phi_1}{t} = \frac{a}{2} G\Theta = \frac{a}{2}\frac{T}{J} = \frac{T}{34a^2 t}$$

$$\tau_1 = \frac{q_1}{2t} = \frac{\phi_1}{2t} = aG\Theta = a\frac{T}{J} = \frac{T}{17a^2 t} \leftarrow \tau_{max}$$

$$\Rightarrow \tau_{max} = \frac{T}{17a^2 t} \le \tau_{yp} \Rightarrow \boxed{T_{yp} = 17a^2 t \tau_{yp}}$$

elastic solution above

3 At T_{yp} : $\phi_1 = \tau_{yp} 2t$ and $\phi_0 = \frac{5}{4}\phi_1 = \frac{5}{2}\tau_{yp} t$

ϕ_1 cannot increase anymore, but ϕ_0 can, until

$$\tau_0 = \frac{\phi_0 - \phi_1}{t} = \tau_{yp} \Rightarrow \phi_0 = \phi_1 + \tau_{yp} t = 3\tau_{yp} t$$

At T_{ult} : $\phi_0 = 3\tau_{yp} t$ and $\phi_1 = 2\tau_{yp} t$

so

$$T_{ult} = 2(4A_1\phi_1 + A_0\phi_0) = 18a^2 t\tau_{yp} \qquad \boxed{T_{ult} = 18a^2 t\tau_{yp}}$$

that is,

$$\boxed{\frac{T_{ult}}{T_{yp}} = \frac{18}{17} = 1.06 \quad \text{or} \quad 6\% \text{ more}}$$

Alternative solution

Because no shear flow along inclined walls $\Rightarrow$ 2 square cells twisting together

$$J_0 = \frac{4A_{in}^2}{\oint \frac{ds}{t}} = \frac{4(a^2)^2}{4\frac{a}{t}} = a^3 t \quad \Leftarrow \text{inner cell}$$

$$J_1 = \frac{4A_{out}^2}{\oint \frac{ds}{t}} = \frac{4(4a^2)^2}{4\frac{2a}{2t}} = 16a^3 t \quad \Leftarrow \text{outer cell}$$

$$\theta = \frac{T_0}{GJ_0} = \frac{T_1}{GJ_1} = \frac{T_0+T_1}{G(J_0+J_1)} = \frac{T}{GJ}$$

T_0: torque taken by inner cell; T_1: torque taken by outer cell

$$\Rightarrow J = J_0 + J_1 = 17a^3 t \qquad \boxed{J = 17a^3 t}$$

Inner cell: $q_0 = \tau_0 t = \frac{T_0}{2A_{in}} = \frac{1}{2\underbrace{A_{in}}_{a^2}}\frac{J_0}{J}T = \frac{T}{34a^2}$

$$\Rightarrow \tau_0 = \frac{T}{34a^2 t}$$

Outer cell: $q_1 = \tau_1 2t = \frac{T_1}{2A_{out}} = \frac{1}{2\underbrace{A_{out}}_{4a^2}}\frac{J_1}{J}T = \frac{2T}{17a^2}$

$$\Rightarrow \tau_1 = \frac{T}{17a^2 t}$$

$$\Rightarrow \tau_{max} = \frac{T}{17a^2 t} \le \tau_{yp} \Rightarrow \qquad \boxed{T_{yp} = 17a^2 t\,\tau_{yp}}$$

For T_{ult}: $\tau_0 = \tau_1 = \tau_{yp}$ $\Rightarrow T_{ult} = 4\left(\underbrace{\tau_{yp}2t\cdot 2a\cdot a}_{\text{out}} + \underbrace{\tau_{yp}ta\cdot\frac{a}{2}}_{\text{in}}\right)$ (torque of each wall)

$$\frac{T_{ult}}{T_{yp}} = \frac{18}{17} = 1.06 \text{ or } 6\% \text{ more} \Leftarrow \qquad \boxed{T_{ult} = 18a^2 t\,\tau_{yp}}$$