ANSWERS

UC Berkeley
Department of Civil Engineering

CE93– QUIZ #2

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Name: ______________________________
SID Number: __________________________

This exam contains 7 pages (including this cover page) and 4 questions. Total of points is 100. All work should be done on the question sheets. You need to show your work to get credit. If you run out of space, use the back of the page to write down your solution. You are allowed to use a calculator and two 5"x 8" cheat sheets. Communication with others is not allowed except with the proctors. No leaving early. If you finish early, again, enjoy solving the bonus problem, or just meditate.
Good luck!

Distribution of Marks

Question	Points	Score
1	10	
2	30	
3	30	
4	30	
Total:	100	

1. (10 points) In recognition of everyone's efforts to study and understand the materials for this exam, y'all get ten (10) free points. Go get the rest!

2. Multiple Choice Questions

   (a) (5 points) Springs in Lebanon tend to be warm, but there is always a probability of rain. Assume that whether or not it rains on a given Spring day is a Bernoulli random variable with probability of rain p = 0.25, and that raining on a given day is independent from raining on another.

   What is the expected value of **non-rainy** days in a week?

   i. 1.31
   ii. 1.75
   iii. 5.25
   iv. 2

   X is binomial

   Mean $= n\underset{\uparrow}{p} = 7(1-0.25) = 5.25$

   non-rainy

   (b) (10 points) Let X and Y be two **independent** random variables. The table below contains some (incomplete) information about the joint PMF of X and Y, as well as their marginals.

-	X = 0	X = 1	X = 2	$P_y(Y)$
Y = 0	0.2			0.4
Y = 1	0.075	0.03	0.045	0.15
Y = 2	0.225	0.09	0.135	0.45
$P_x(X)$	0.5	0.2 a	0.3	1.0

   What is the value of "a" in the table?

   i. 0.4
   ii. 0.15
   iii. 0.30
   iv. 0.2

   Independent

   $p(a) = P(X=1)$

   $P(X=1, Y=2) = P(X=1)\,P(Y=2)$

   $0.09 = a\,0.45 \rightarrow a = 0.2$

   What is the value of P(Y=1|X=2)?

   i. 0.045
   ii. 0.6
   iii. 0.075
   iv. 0.15

   $P(Y=1|X=2) = \frac{P(Y=1, X=2)}{P(X=2)} = \frac{0.045}{0.30}$

   $= 0.15$

   OR using independence (easier!)

   $P(Y=1|X=2) = P(Y=1)$

(c) (10 points) A car rental company receives bookings for SUVs, full size, and medium size cars. Assume that the types of bookings received are independent, and that for any given booking, the probability of the customer requesting an SUV is p = 0.25

The company opens on a Monday. What is the expected number of bookings received before getting an SUV booking?

i. 4
ii. 3
iii. 0.5
iv. 0.25

Geometric

X = # times until first success

$\mu_x = \frac{1}{p} = 4$

$\underbrace{000}_{\text{3 before}}1$ book

What is the average return period, or the average number of bookings between two SUV bookings?

from lab

i. 3
ii. 4
iii. 2
iv. 0.5

1 0 0 0 1

book book

return period

(d) (5 points) Joan's daily commute distance is a random variable, X, with an average of 3 miles and a standard deviation of 0.5 miles. Hassan's daily commute distance is another random variable, Y, with an average of 2 miles and a standard deviation of 0.2 miles. Assume that Joan's and Hassan's commute distances are independent. Let W be another random variable that is the total daily commute distance of Joan and Hassan (combined).

What is the standard deviation (in miles) of W?

i. 0.70
ii. 0.54
iii. 0.40
iv. 0.29

$\mu_X = 3$ $\mu_Y = 2$

$\sigma_X = 0.5$ $\sigma_Y = 0.2$

$W = X + Y$

$\sigma_W = \sqrt{\sigma_X^2 + \sigma_Y^2}$

$= \sqrt{0.5^2 + 0.2^2}$

$= \sqrt{0.29}$

$= 0.54$

3. Two random variables, X and Y, have the following joint PDF:

$$f_{X,Y}(x,y) = 3x^2(2-2y)$$

$$0 \le x \le 1$$

$$0 \le y \le 1$$

(a) (15 points) Calculate the marginal probabilities of X and Y, $f_X(x)$ and $f_Y(y)$. Are X and Y independent? Why or why not?

$$f_X(x) = \int_{y=0}^{1} f_{X,Y}(x,y)\,dy = \int_{y=0}^{1} 3x^2(2-2y)\,dy = 3x^2(2y-y^2)\Big|_{y=0}^{1}$$

$$\Rightarrow f_X(x) = 3x^2(2-1-0) \Rightarrow \boxed{f_X(x) = 3x^2}$$

$$f_Y(y) = \int_{x=0}^{1} f_{X,Y}(x,y)\,dx = \int_{x=0}^{1} 3x^2(2-2y)\,dx = x^3(2-2y)\Big|_{x=0}^{1}$$

$$\Rightarrow \boxed{f_Y(y) = 2-2y}$$

Independent? $f_X(x) \cdot f_Y(y) = 3x^2(2-2y) = f_{X,Y}(x,y)$

Thus, X & Y are independent

(b) (15 points) Derive an expression for the joint cumulative distribution function (CDF) of X and Y, $F_{X,Y}(x,y)$. What is the probability that X < 0.5 and Y < 0.5?

$$F_{X,Y} = \int_{x=0}^{x}\int_{y=0}^{y} f_{X,Y}(x,y)\,dy\,dx = \int_{x=0}^{x}\int_{y=0}^{y} 3x^2(2-2y)\,dy\,dx$$

$$= \int_0^x 3x^2 \times \left[2y-y^2\right]_{y=0}^{y} dx = \int_0^x 3x^2(2y-y^2)\,dx$$

$$= (2y-y^2)\cdot\left[x^3\right]_{x=0}^{x} \Rightarrow \boxed{F_{X,Y}(x,y) = (2y-y^2)x^3}$$

$$P(X<0.5, Y<0.5) = F_{X,Y}(x=0.5, y=0.5) = (2\times 0.5 - 0.5^2)\,0.5^3$$

$$\Rightarrow \boxed{P(X<0.5, Y<0.5) = 0.09375}$$

4. Based on historical data, we know that a major flood hits the city of Beirut on average once every 15 years. Assume that the occurrence of major flooding events follows a Poisson distribution.

   (a) (15 points) What is the probability of one flood hitting Beirut in the next year? What is the expected value of the number of floods hitting Beirut in the next 5 years?

$$P(X=1) = \frac{e^{-\frac{1}{15}\times 1} \times \left(\frac{1}{15}\times 1\right)^1}{1!} = 0.062$$

$$E\left(\text{floods } 5\text{ year}\right) = \lambda \times t = \cancel{4}\ \frac{1}{15}\times 5 = \cancel{5}\ \frac{1}{3}$$

   (b) (15 points) Now assume that the occurrence of a flood in a given year is a Bernoulli random variable, where the probability of success (a flood happening in a given year) is the one you calculated in part (a). A Civil engineer decides to design her house to withstand such extreme weather events. To do so, she comes up with a concept that has a design lifetime of 10 years, and that is robust enough to withstand a flood with a probability of 0.99, when a flood happens.

   What is the probability that her house does not fail during its lifetime? Assume that floods are the only reason her house may fail. (Note: If you did not get an answer for part (a), assume a value for the probability and continue accordingly).

Prob. of flood in a year = $P = 0.062$ (from part a)

$P_{\text{no damage} \mid \text{Flood}} = 0.99$

$$P(\text{no damage in a year}) = P(\text{nodamage} \mid \text{Flood}) \times P(\text{Flood}) + P(\text{no Flood})$$
$$= 0.99 \times 0.062 + (1 - 0.062)$$
$$= 0.9999$$

$$P(\text{no damage in 10 years}) = (0.999)^{10} = 0.9999.$$

(c) (Bonus) Given that no floods hit Beirut in the last 15 years, show that the probability of no floods happening in the next 15 years remains the same.

Let $N(t)$ be the # of floods in time t.

We want to prove that $P(N(30)=0 \mid N(15)=0) = P(N(15)$

in other words, we want to prove that the probability of no earthquakes in 30 years, given that no earthquake happen in 15 years, is simply the probability of no earthquakes in 15 years.

$$P(N(30)=0 \mid N(15)=0) = \frac{P(N(30)=0, \overbrace{N(15)=0}^{\text{redundant}})}{P(N(15)=0)}$$

$$= \frac{P(N(30)=0)}{P(N(15)=0)}$$

$$= \frac{e^{-30\lambda}}{e^{-15\lambda}}$$

$$= e^{-30\lambda + 15\lambda}$$

$$= e^{-15\lambda}$$

$$\therefore \boxed{P(N(30)=0 \mid N(15)=0) = P(N(15)=0)}$$