UC Berkeley

Department of Civil and Environmental Engineering

CE93 Fall 2022
Midterm 1

Instructor: Aakash Bangalore Satish
GSI: Joyce Chen
Reader: Lixin Pan

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Name: ______________________________

SID Number: ______________________________

- This exam contains 10 printed pages (including the cover page). Write your name and student ID number on all pages of the question paper and your cheat sheet.
- You are allowed to use a calculator and a 2-sided cheat sheet of size 8.5 x 11 inches. Submit your cheat sheet with the exam booklet to score 10 points on the exam.
- Please show all steps you took to reach the answer. Points will be liberally awarded for each correct step towards the solution. If you need more space than provided, please use the back side of the sheets, and clearly write the question number corresponding to your work on the back pages, if any.
- Please wait till the end of the exam to leave.

Question	**Points**	**Score**
1	22	
2	20	
3	26	
4	12	
Cheat sheet	10	
Total	**90**	

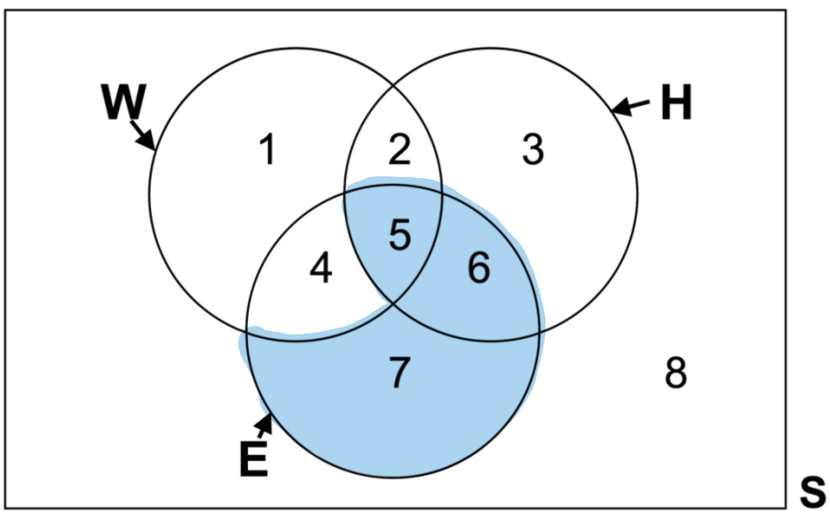
Question 1: (22 points)
The engineer of a landfill containment system believes that leakage of contaminants from a landfill will happen "during extremely heavy rainfall, and either the clay was not well compacted or there were holes in the geomembrane" (event A).
Leakage could also occur "under ordinary rainfalls (i.e., without extremely heavy rainfall), but only when the clay was not well compacted and the geomembrane contained holes" (event B).
Let

W = event of well-compacted clay
H = event of geomembrane containing holes
E = event of extremely heavy rainfall

(a) Express event B symbolically in terms of the events W, H, and E defined above. (**2 points**)

$B = \overline{E} \cap \overline{W} \cap H$

(b) Represent event A in the following Venn diagram. (**2 points**)



$A = E \cap (\overline{W} \cup H)$ (given)

$= (E \cap \overline{W}) \cup (E \cap H)$ (distributive property of intersection over union)

$= (6 \cup 7) \cup (5 \cup 6)$

$= 5 \cup 6 \cup 7$

Given that

W (event of well-compacted clay) occurs with 90% probability $\rightarrow P(W) = 0.9$

H (event of geomembrane containing holes) is 30% likely $\rightarrow P(H) = 0.3$

E (event of extremely heavy rainfall) has a likelihood of 20%. $\rightarrow P(E) = 0.2$

If the geomembrane contains holes, this might indicate poor quality of workmanship and the probability that the clay is well-compacted is reduced to 60%. $\rightarrow P(W|H) = 0.6$

With this information,

(c) Determine the probability of the event $(\bar{W} \cup H)$. (**10 points**)

$P(\bar{W} \cup H) = P(\bar{W}) + P(H) - P(\bar{W} \cap H)$ (Addition rule)

$P(\bar{W}) = 1 - P(W) = 1 - 0.9 = 0.1$ (Probability of complement of an event)

$P(\bar{W} \cap H) = P(\bar{W}|H)P(H)$ (Multiplication rule)

$P(\bar{W}|H) = 1 - P(W|H) = 1 - 0.6 = 0.4$ (Conditional probability of complement of an event)

$\therefore P(\bar{W} \cap H) = 0.4 \times 0.3 = 0.12$

& $P(\bar{W} \cup H) = 0.1 + 0.3 - 0.12 = \underline{\underline{0.28}}$

Additionally, the compaction of clay or the occurrence of holes in the geomembrane obviously do not influence the amount of future rainfall (i.e., event E is statistically independent of the event W, event E is statistically independent of event H).

(d) Determine the probability of the event $\bar{E} \cap (\bar{W} \cap H)$. (**8 points**)

$P(\bar{E} \cap (\bar{W} \cap H)) = P(\bar{E})P(\bar{W} \cap H)$ (since E is statistically independent of W or H)

$P(\bar{E}) = 1 - P(E) = 1 - 0.2 = 0.8$

$P(\bar{W} \cap H) = 0.12$ (as calculated in part (c))

$\therefore P(\bar{E} \cap (\bar{W} \cap H)) = 0.8 \times 0.12 = \underline{\underline{0.096}}$

Question 2: (20 points)

The total load, X (in tons) on the roof of a building has the following probability density function:

$$f_X(x) = \begin{cases} c/x^3 & 3 \le x \le 6 \\ 0 & elsewhere \end{cases}$$

Hints: $\int x^n \, dx = \frac{1}{n+1} x^{n+1}$ if $n \neq -1$

$\int x^{-1} dx = \ln x$

(a) What is the value of the constant c? (**6 points**)

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$ (property of PDF, this follows from the axiom of normalization)

$$\Rightarrow \int_{-\infty}^{3} 0\,dx + \int_{3}^{6} \frac{c}{x^3}\,dx + \int_{6}^{\infty} 0\,dx = 1$$

$$\Rightarrow c\int_{3}^{6} x^{-3}\,dx = 1$$

$$\Rightarrow c\left[-\frac{1}{2}\frac{1}{x^2}\right]_3^6 = \frac{-c}{2}\left[\frac{1}{6^2} - \frac{1}{3^2}\right] = \frac{c}{24} = 1$$

$$\Rightarrow \underline{\underline{c = 24}}$$

(b) What is the expected total load? (**6 points**)

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$ (by definition of mean/expected value for a continuous RV)

$$= \int_{-\infty}^{3} x \cdot 0\,dx + \int_{3}^{6} x \cdot \frac{c}{x^3}\,dx + \int_{6}^{\infty} x \cdot 0\,dx$$

$$\Rightarrow E(X) = c\int_{3}^{6} \frac{1}{x^2}\,dx = c\left[-\frac{1}{x}\right]_3^6 = c\left[-\frac{1}{6} + \frac{1}{3}\right] = \underline{\underline{\frac{c}{6}}} \text{ tons}$$

$$\therefore E(X) = 4 \text{ tons}$$ (since $c = 24$)

(c) Suppose the roof can carry only 5.5 tons before collapse. What is the probability that the roof will collapse? (**8 points**)

The roof will collapse if the total load on the roof is greater than 5.5 tons (i.e. if $X > 5.5$)

$$P(\text{collapse}) = P(X > 5.5) = 1 - P(X \leq 5.5) = 1 - F_X(5.5)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du \quad \text{(by definition)}$$

If $x < 3$, $F_X(x) = 0$ (since $f_X(x) = 0 \;\forall x < 3$).

If $3 \leq x \leq 6$, $F_X(x) = F_X(3) + \int_3^x f_X(u)\,du = c\int_3^x \frac{1}{u^3}\,du = \frac{-c}{2}\left[\frac{1}{9} - \frac{1}{x^2}\right]$

$\therefore F_X(5.5) = \frac{-c}{2}\left[\frac{1}{9} - \frac{1}{(5.5)^2}\right] \simeq 0.03903\,c \simeq 0.9366$ (since $c=24$)

$\therefore P(\text{collapse}) \simeq \underline{\underline{0.06336}}$

(d) What is the coefficient of variation of the total load? (10 points – ***BONUS QUESTION****)*

Coefficient of variation, $\delta_X = \frac{\sigma_X}{|\mu_X|}$ (by definition)

$\sigma_X = \sqrt{E(X^2) - (\mu_X)^2}$ (by definition)

$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$ (by definition)

$$= c\int_3^6 \frac{1}{x}\,dx = c\left[\ln(x)\right]_3^6 = c\left[\ln(6) - \ln(3)\right]$$

$\simeq 0.6931\,c = 16.6344$ (since $c=24$)

$\therefore \sigma_X \simeq \sqrt{16.6344 - 16} \simeq 0.7965$

$\therefore \delta_X \simeq \frac{0.7965}{4} = \underline{\underline{0.1991}}$

Question 3: (26 points)

The joint probability mass function of precipitation, X (inches) and runoff Y (cubic feet per second) (discretized here for simplicity) due to storms at a given location is as follows:

	$X = 1$	$X = 2$	$X = 3$
$Y = 10$	a	0.15	0.0
$Y = 20$	$2a$	0.25	$5a$
$Y = 30$	0.0	0.10	$2a$

(a) What is the value of the constant a in the table? (**4 points**)

$$\sum_x \sum_y P_{X,Y}(x,y) = 1$$ (property of joint PMF, this follows from the axiom of normalization)

$$\Rightarrow 10a + 0.5 = 1$$

$$\Rightarrow a = \underline{\underline{0.05}}$$

(b) What is the probability that the next storm will bring a precipitation of less than 2 inch and a runoff of more than 20 cubic feet per second? (**2 points**)

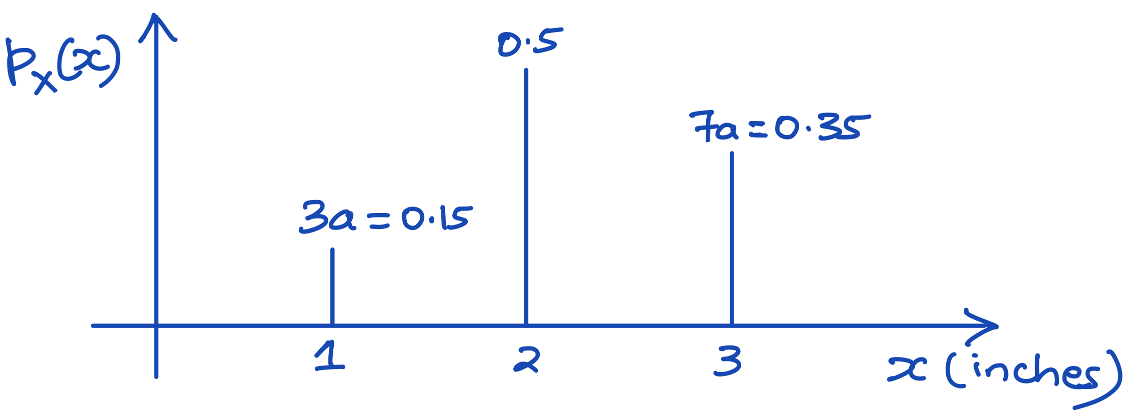
$$P(X<2, Y>20) = P(X=1, Y=30) = \underline{\underline{0.0}}$$

(For the discrete RV X defined in the question, the event $X<2$ is equal to the event $X=1$
similarly, for the given RV Y, the event $Y>20$ is equal to the event $Y=30$)

(c) Determine the marginal probability mass function of precipitation. Plot the marginal probability mass function of precipitation. (**8 points**)

$$p_X(x) = \sum_y p_{X,Y}(x,y) \quad \text{(by definition)}$$

x	1	2	3
$p_X(x)$	$3a$	0.5	$7a$



(d) Determine and plot the probability mass function of runoff for a storm whose precipitation is 2 inches. What is the expected runoff for a storm whose precipitation is 2 inches? (**12 points**)

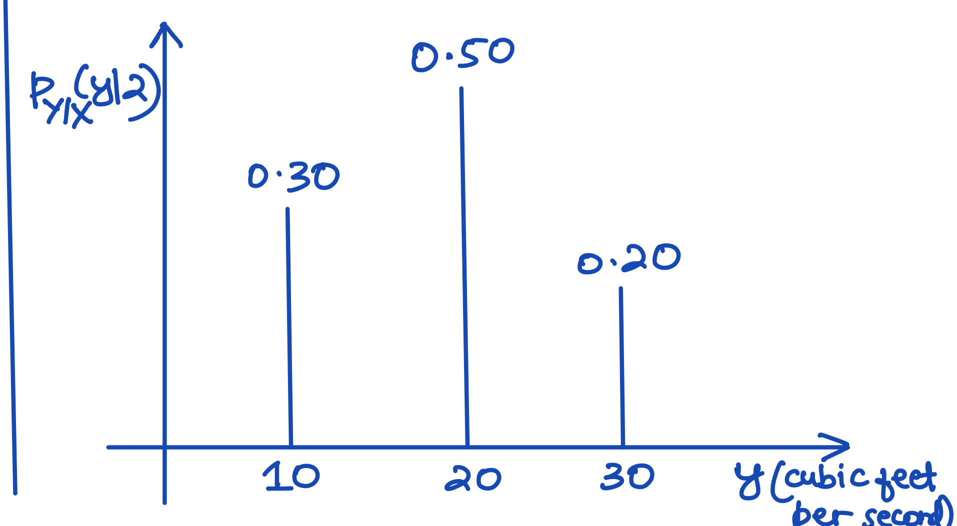
$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \quad \text{(by definition)}$$

Given $X=2$, $p_X(2) = 0.5$ (as calculated in part (c))

$$\therefore p_{Y|X}(y|2) = \frac{p_{X,Y}(2,y)}{0.5}$$

$\Rightarrow$

y	$p_{Y\|X}(y\|2)$
10	0.30
20	0.50
30	0.20



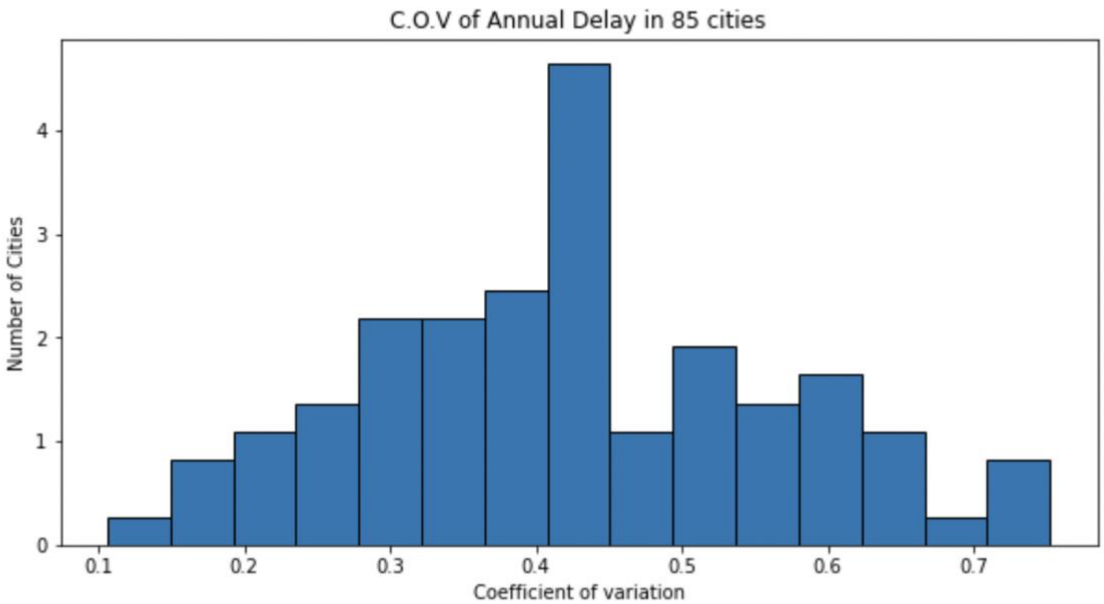
$$E(Y|x) = \sum_y y\, p_{Y|X}(y|x) \quad \text{(by definition of conditional PMF)}$$

$$\therefore E(Y|2) = \sum_y y\, p_{Y|X}(y|2) = 10\times0.3 + 20\times0.5 + 30\times0.2 = \underline{\underline{19}}$$

Question 4: (12 points)

(a) Given a dataset which contains the counts of coefficient of variation (C.O.V) in travel delay in 85 cities, which of the lines of code should replace line 2 in the following code snippet to produce the **density histogram** shown in the figure? (**4 points**)

```
1 fig, ax = plt.subplots(figsize=(10,5))
2 ax.hist(...)
3 ax.set_title('C.O.V of Annual Delay in 85 cities')
4 ax.set_ylabel('Number of Cities')
5 ax.set_xlabel('Coefficient of variation')
6 plt.show()
```




```
# Option A
ax.hist(cov_annual_delay, bins=15, density=False, cumulative=True, ec='black')
```

```
# Option B
ax.hist(cov_annual_delay, bins=15, density=True, cumulative=False, ec='black')
```

```
# Option C
ax.hist(cov_annual_delay, bins=18, density=True, cumulative=False, ec='black')
```

```
# Option D
ax.hist(cov_annual_delay, bins=18, density=False, cumulative=False, ec='black')
```

(b) Given a numpy array called "delays" containing data of average annual travel delay in hours, which of the following is the correct code to calculate the **probability** that average annual travel delay is higher than 8 hours and less than 24 hours? (**4 points**)

```
# Option A
probability = sum(delays>8.0 & delays <24.0)/len(delays)
```

```
# Option B
probability = (delays>8.0 | delays<24.0)/len(delays)
```

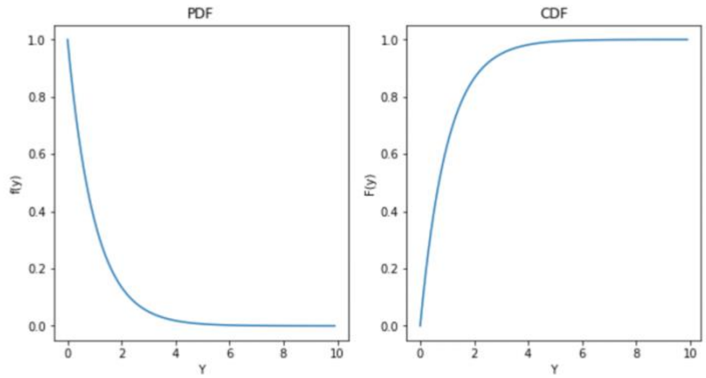
```
# Option C
probability = sum((delays>8.0) & (delays<24.0))/len(delays)
```

```
# Option D
probability = sum((delays>8.0) | (delays <24.0))/delays
```

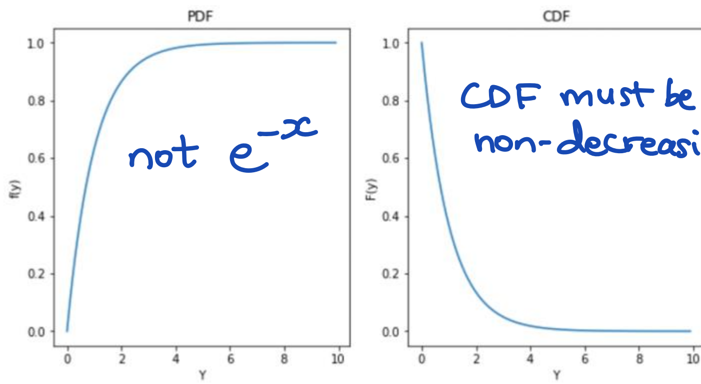
Which of the following figures is generated by the code snippet on the right? (**4 points**)

```python
def exponential_pdf(x):
    return np.exp(-x)

def exponential_cdf(x):
    return 1-np.exp(-x)

Y = np.arange(0, 10, 0.1)

plt.figure(figsize=(10,5))
plt.subplot(121)
plt.plot(Y, exponential_pdf(Y))
plt.title('PDF')
plt.xlabel('Y')
plt.ylabel('f(y)')

plt.subplot(122)
plt.plot(Y, exponential_cdf(Y))
plt.title('CDF')
plt.xlabel('X')
plt.ylabel('F(y)')

plt.show()
```

a)



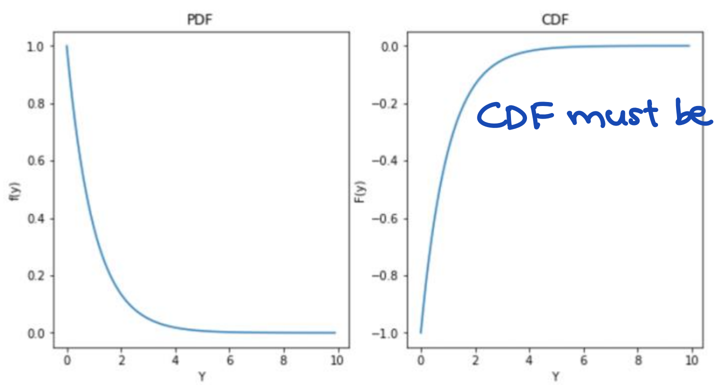
b)



not e^{-x}

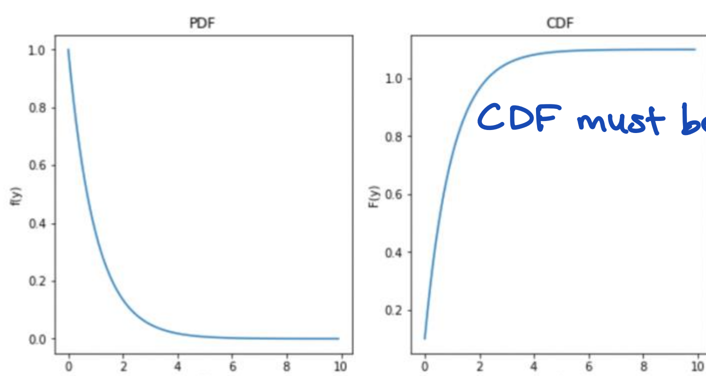
CDF must be non-decreasing

c)



CDF must be ≥ 0

d)



CDF must be ≤ 1